On applications of graph/network theory to problems in communication systems

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ABSTRACT

Graph/network theory results are applicable to problems in communications. As a representative example, the node coloring problem in graph theory is applicable to the channel assignment problem in cellular mobile communication systems. The node coloring problem is NP-complete, meaning that optimally solving it is very difficult. Therefore, we use heuristic algorithms for the channel assignment problem. In this case, the graph theory results show the legitimacy of using heuristic techniques. On the other hand, we can directly apply graph theory to communication problems. For example, on contents delivery services in the Internet, we place mirror servers that provide the same contents on the network. Location problems on flow networks are applicable to mirror server allocation problems. In a simple case, we can efficiently solve the problem. In this paper, we concentrate on multi-hop wireless networks and consider the relationship between their problems and the results of graph/network theory.

Keywords: Graph Theory, Multi-hop Wireless Networks, Coloring Problem, Location Problem, Network Coding, Routing Protocol

1. INTRODUCTION

Tremendous growth in the popularity of mobile communication services has occurred worldwide. As a result, in our small spheres of everyday living, we are never far from mobile communication services. Communication networks, including mobile ones, are becoming a new social infrastructure in many countries. Moreover, IT-based societies are expected to be established to improve human life in many countries. However, many challenging problems remain in the area of communications, especially multi-hop wireless networks (in other words, mobile ad hoc networks) to realize the full establishment of IT-based societies.

Graph/network theory is applicable to problems in communications, including multi-hop wireless networks. As a representative example, the node coloring problem in graph theory is applicable to the channel assignment problem in cellular mobile communication systems. In this paper, after introducing the node coloring problem with an application to channel assignment problems, we introduce relationships between the problems on multi-hop wireless networks and graph/network theory.

2. SHORT HISTORY OF GRAPH/NETWORK THEORY CLOSELY RELATED TO TOPICS OF THIS PAPER

An electrical circuit is a typical example of a network, and the connection structure of such circuit elements as resistors, capacitors, inductors, and/or voltage sources is a typical example of a graph. In 1845, G. R. Kirchhoff introduced KCL and KVL equations [1] to prove the equilibrium of the Wheatstone bridge made in 1843, where KCL and KVL are the acronyms for Kirchhoff's Current Law and Kirchhoff's Voltage Law. The KCL equations were given as an *a priori* theorem, i.e., without proof from the knowledge of electromagnetism, and the KVL equations were proven based on the knowledge that the voltage of a circuit element is defined as the difference of the node potentials of the element's end-nodes. In 1847, Kirchhoff [2] formulated a system of equations of an electrical circuit consisting of resistors and voltage sources, and then completely solved the system of equations. Until now, after the Kirchhoff's method of circuit analysis, various methods of circuit analysis such as nodal analysis, loop analysis, mesh analysis, cutest analysis, mixed analysis, modified nodal analysis, and so on, have been developed for more general electrical circuits, including electronic analog circuits.

On the other hand, O. Veblen [3] pointed out that H. Poincare constructed his theory of combinatorial topology after being inspired by Kirchhoff's 1847 paper. Poincare's theory in the case of 1-dimension relates to a graph theory that uses the concepts of an incidence matrix, a fundamental cutset matrix, and a fundamental loop matrix. KCL has been concep-

tually extended to the conservation law of flows, and KVL has been conceptually extended to the equilibrium law of tensions. The theory of flows and/or tensions of networks has been largely developed with applications to various networks since Ford and Fulkerson's [4]. Until now, in the theory of flows and/or tensions of networks, efficient algorithms for finding the shortest paths, minimal trees, maximal flows, minimum cost flows, node (or edge) disjoint paths with minimum total length subject to node-disjointness between two nodes and others have been extensively developed.

3. MULTI-HOP WIRELESS NETWORKS

Multi-hop wireless networks [5] are autonomous systems of mobile nodes connected by wireless links and consist of nodes and wireless links. The nodes are classified into two kinds: one is a mobile terminal such as mobile phone, personal computer, handheld computer, Personal Digital Assistant (PDA) and so on, each of which has a router communication function for other terminals, and the other is a special terminal that acts only as a router for mobile terminals and has a long-life or a permanent battery. Such special terminals (called service terminals) are distributed or located randomly or intentionally beforehand, but in few numbers, where many mobile terminals are gathered. Each wireless link, which is an ad hoc communication link constructed by wireless networking based on the requests of mobile terminals, has an indefinite limited life-time of connection depending on its terminal nodes' various situations. . In a multi-hop wireless network, two nodes can directly communicate with each other if they are close enough, or they can communicate with each other through one or more intermediate nodes that double as routers if they are not within radio range (Fig.1).
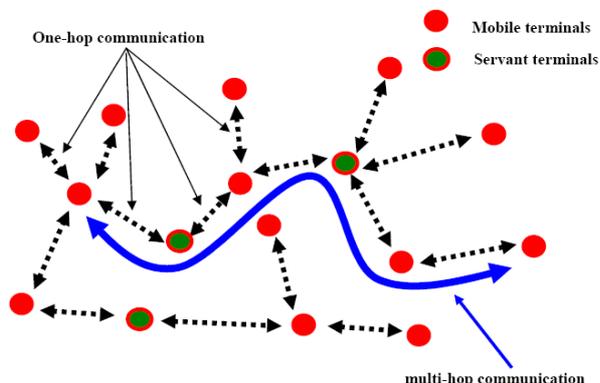

Fig.1: *Multi-hop wireless networks*

Such multi-hop wireless networks are useful in various applications when cellular infrastructure is neither available nor realistic or when ubiquitous communication services are required without the presence or use of a fixed infrastructure:
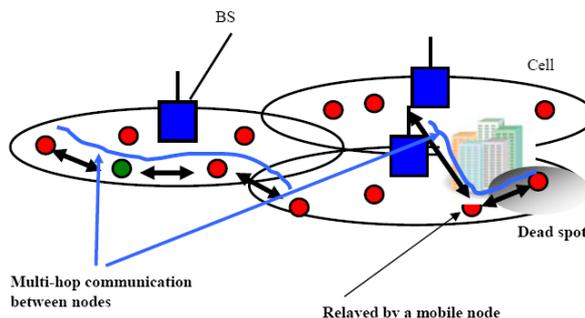

Fig.2: *Multi-hop wireless networks and cellular systems*

*Co-operative existence of multi-hop wireless networks with a cellular network (Fig. 2)
*Vehicles to vehicle networks
*Mobile robot networks
*Sensor networks
*On-the-fly conferencing applications
The realization of such multi-hop wireless networks is based on routing protocols and relay functions to each node.

4. PRELIMINARIES

Here we introduce graphs and some problems in graph theory. Let G=(V,E) be a graph with node set V and edge set E. A node coloring of graph G is a map c:V → S ={1,2,...,k} such that $c(v_1) \neq c(v_2)$, whenever nodes v_1 and v_2 are adjacent. The elements of set S are called the available colors. The only interesting aspect of S is its size. Smallest integer k such that node coloring c:V → S={1,2,...,k} is called the chromatic number of graph G. If the chromatic number of G is k and h ≥ k, G is said to be h-colorable (Fig. 3).
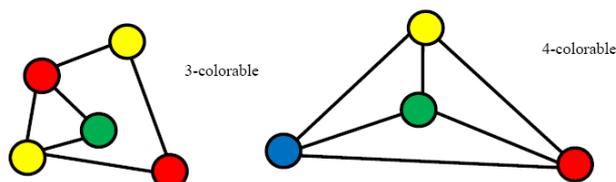

Fig.3: *Node colorings of graphs*

Next we explain an efficient algorithm, which requires a number of steps that grows as a polynomial in the size of the input; it is also called a polynomial algorithm. Efficient algorithms have been applied to fundamental problems in such fields as communication networks including multi-hop wireless networks, computer networks, VLSI circuit design technologies, and so forth.

A problem for which no efficient algorithm is known is said to be NP-complete. The class of NP-

complete problems has the following interesting properties:

1) No NP-complete problem can be solved by any known polynomial algorithm.

2) If there is a polynomial algorithm for any NP-complete problem, then there are polynomial algorithms for all NP-complete problems.

Based on these two facts, there is conjecture that no polynomial algorithm exists for any NP-complete problem. However, this conjecture hasn't been proved yet. In worst cases, any algorithm that correctly solves NP-complete problems will require an exponential amount of time and hence will be impractical for all but very small instances. Once a problem is known to be NP-complete, one is generally willing to settle for less ambitious goals than developing an algorithm that always finds an exact solution and may choose one of several possible alternative approaches to the problem. NP-completeness provides the legitimacy of the use and development of heuristic techniques, or approximation algorithms for solving an NP-complete problem. For example, the node coloring problem of a graph is an example of a NP-complete problem [6]. The node coloring problem asks, "Is it possible to color the nodes of graph G with h colors?" This problem is NP-complete for any h≥3. Even the problem restricted to h = 3 for planar graphs of maximal degree 4 is still NP-complete. Although it was proved that any planar graph is 4-colorable [7], there is presently no efficient algorithm to color the nodes of any planar graph with four colors.

Apart from a node coloring of G=(V,E), an edge coloring of G exists that is defined as a map $c:E \rightarrow S$ with $c(e_1) \neq c(e_2)$ for any adjacent edges e_1 and e_2. The smallest integer k for which an edge coloring $c:E \rightarrow \{1,2,\ldots,k\}$ exists is the chromatic index of G. This problem is also NP-complete.

5. RECENT TOPICS

5.1 Channel assignment problem in cellular mobile communication systems

In a cellular mobile communication system [8], the service area is divided into many small cells and each channel is simultaneously re-used in some cells (Fig. 4). We assume that if a cell is adjacent to another cell, we can assign different channels to the two cells. This assumption resembles map coloring, which means that pairs of adjacent countries have different colors on maps. Here, we construct a graph as follows. Each node represents a country, and if two countries are adjacent, we set an edge between the corresponding two nodes. In this case, the channel assignment problem is equivalent to the node coloring problem [9]. Since the problem is NP-complete in general, we cannot expect the algorithm to efficiently obtain the optimal solution. Therefore, we

have to use or develop a heuristic algorithm. A solution obtained by any algorithm of the problem is not always optimal. A problem of finding an optimal channel assignment in cellular mobile communication systems can be modeled as a problem of coloring of a graph. Hence, the optimal channel assignment problem cannot be solved efficiently because of its NP-completeness. So the existing heuristic techniques for channel assignment have been developed because of the legitimacy from the NP-completeness.
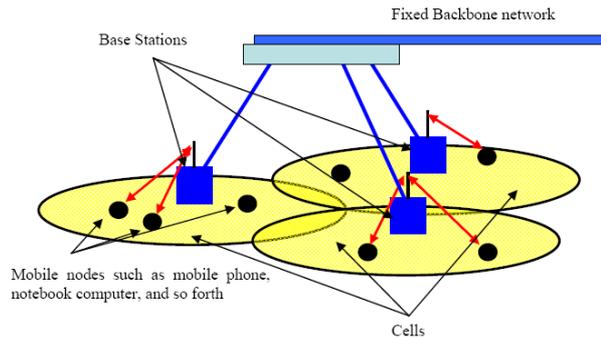

Fig.4:   *Cellular mobile communication system*

Here we consider a more appropriate model to solve the problem [10] that considers the degree of co-channel interference. If the number of edges between two nodes u_1 and v_1 is one and the number of edges between nodes u_2 and v_2 is k, the degree of interference between u_2 and v_2 is k times of the degree of the interference between u_1 and v_1. We define a generalized coloring problem as follows. We assign colors to nodes in such a way that, for each node v, the number of identically colored nodes adjacent to v is not larger than k. Fig. 5 shows an example of this coloring (k=2) because the black nodes are not adjacent and the maximum number of white adjacent nodes is two. In the case of k = 0, this coloring is equivalent to the conventional node coloring.
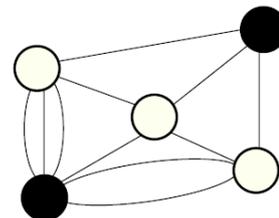

Fig.5:   *Generalized node coloring*

5.2 Channel assignment problems in multi-hop wireless networks

For multi-hop wireless networks, we assign channels to communication between two terminals. This is a kind of edge coloring problem. For conventional edge coloring, we assign different colors to adjacent
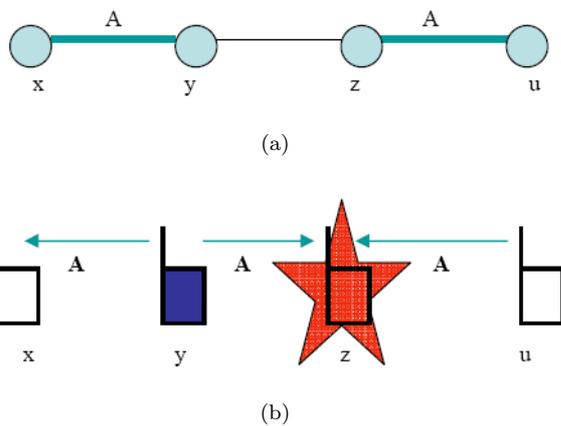
(a)

(b)

Fig.6: *(a) Conventional edge coloring (b) Multi-hop wireless network corresponding to Fig. 6(a)*

edges. However, in this model, co-channel interference may occur. In Fig. 6(a), channel A is assigned to edge (x,y) and (z,u). Fig.6(b) represents the assignment in a multi-hop wireless network. In Fig. 6(b), terminal y sends data in all directions when y communicates with x. Since terminal u communicates with z using channel A, terminal z cannot receive carrier from terminal u.

Therefore, we defined the following edge coloring and called it strong edge coloring. A strong edge coloring of G is an assignment of colors to the edges of G such that every two edges of distance at most two receive different colors. The distance of two edges is at most two means the edges are adjacent, or there is an edge between them. Fig.7 is an example of strong edge coloring. A letter on each edge represents an assigned color. The problem of finding a strong edge coloring with a minimum number of colors for general graphs is NP-complete [11].
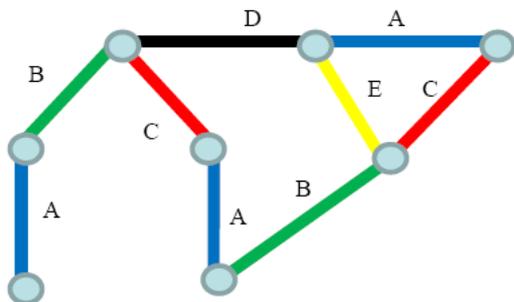

Fig.7: *Strong edge coloring*

We consider a more appropriate edge coloring model [12] that also takes the degree of co-channel interference into consideration. If node v is adjacent to u_1 and u_2, and the number of edges between v and u_1 is one and the number of edges between v and u_2 is k, the degree of the interference from u_2 in v is k times of the degree of interference from u_1 in v.

For example, in Fig. 8, a color is assigned to edge (x,y) and (z,u). Terminal z receives a carrier from terminal u and receives co-channel interference from y. The number of edges between z and u is two, and the number of edges between z and y is one. Therefore, the carrier-to-interference ratio (CIR) is 2/1.

Fig.8: *Edge coloring*

Next we define a generalized edge coloring problem as follows. We assign colors to edges in such a way that, for each edge e, the carrier-to-interference ratio (CIR) is not less than k. In the case of $k = \infty$, this coloring is equivalent to strong edge coloring. Solving this problem is very hard. Even for a tree, this edge coloring problem is NP-complete, but a strong edge coloring problem can be solved in polynomial time.

5.3 Mirror server location problem in networks

Location theory [13] on networks is concerned with selecting the best location in a specified network for facilities. As a representative example, we assign mirror servers to nodes in the following condition called the mirror servers' condition:

*Since each non-mirror node can access the mirror servers with k-edge disjoint paths.

Under the condition, each node secures communication with servers even if some edges break down. The mirror servers' location problem in networks must assign the servers to nodes to minimize the number of assigned servers. Fig. 9 solves the mirror servers' location problem in networks with k=3. 3-edge disjoint paths from node v to mirror servers are shown in Fig. 10, for example.
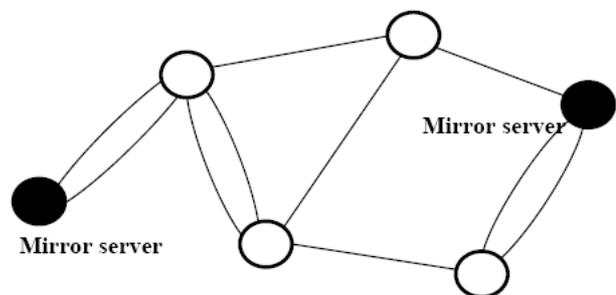

Fig.9: *Solution of mirror servers' location problem in networks*

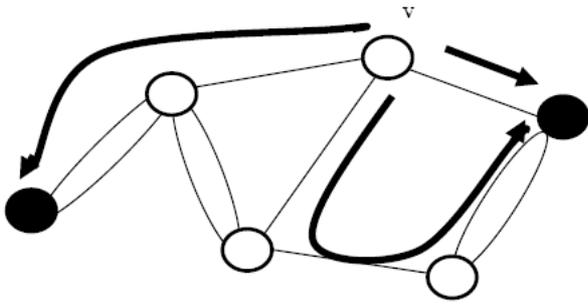A solution can be obtained in the following simple algorithm [14]:

Fig.10: *3-edge disjoint paths from v to mirror servers*

V={v_1,v_2,...,v_n}:node set of graph G
Output: S (solution to this problem)
Step 1: S=V, i=1
Step 2: If S-{v_i} satisfies the mirror servers'
           condition, then S=S-{v_i}.
Step 3: If i=n, then end, else i=i+1 and go to
           Step 2.
In this case, we can directly use this result in graph theory. See [15] for further study.

5.4 Network coding and multi-hop wireless networks

Network coding [16] is a new architecture for wireless networks, and various applications are expected with it. As a representative example, we consider the communication between nodes u and v in Fig. 11. In a conventional approach, four steps are needed (Fig. 12(a)). However, with the concept of network coding, we can reduce it to three steps (Fig. 12(b)) because the packet from node w uses the EX-OR of the packets from u and v, and each node can decode data from the received packet [17]. Efficient communications with network coding are expected, however, presently little concrete indication exists. A graph theoretical approach to this problem can be found in [18].
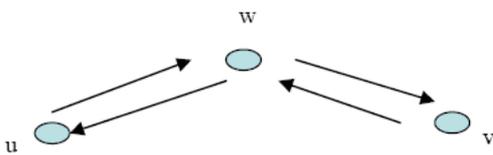

Fig.11: *Communication between vertices u and v*

5.5 Two-node-disjoint path routing in multi-hop wireless networks

In a multi-hop wireless network, a routing protocol is executed to find at least a multi-hop path between a source node and a destination node. There have
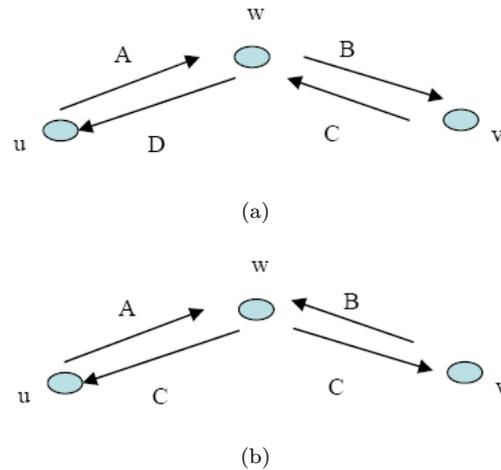

(a)

(b)

Fig.12: *(a) Conventional approach (b) Approach with network coding*

been many proposals for routing protocols, which are roughly classified into two kinds: reactive and proactive [5]. Reactive routing protocols only begin to make a routing table when a request for the transmission of a packet arrives at a source node. This approach reduces overhead to learn the network's topology information. Reactive routing protocols include Dynamic Source Routing (DSR) [19], Ad hoc On-Demand Distance Vector (AODV) [20], and so on. Proactive routing protocols prepare a routing table in advance by exchanging control messages between mobile nodes and updating the table if the network topology changes. Proactive routing protocols include Destination-Sequenced Distance-Vector (DSDV) [21].

Each of the above routing protocols constructs a multi-hop path between a source and a destination; however, we often encounter a situation where source and destination nodes share more than one multi-hop path. Furthermore, these can sometimes be disjoint paths. The introduction of disjoint paths is effective on load balancing, fault tolerance, and so on. Hence, multipath routing protocols have been proposed to construct disjoint paths in multi-hop networks. Split Multipath Routing (SMR) [22] is one multipath routing protocol that constructs two disjoint paths in an on-demand fashion. In SMR, the destination selects two routes: one is the shortest delay route, and the other is maximally disjoint to the first route. An efficient algorithm of finding node disjoint paths as well as an efficient algorithm of finding edge-disjoint paths with minimum total length between two nodes of a network was given in 1974 by J.W. Suurballe [23], and as a revised version of the algorithms a quick algorithm for finding shortest pairs of disjoint paths was given by J.W. Suurballe [23] and R.J. Tarjan [24] in 1984. We can apply the Suurballe-Tarjan algorithm to finding two disjoint paths in a multi-hop wireless network, if they exist, that minimize the total

number of hops in both of the paths. One of noteworthy examples of the application of the Suurballe-Tarjan algorithm, there is a novel polynomial time algorithm that optimally solves the minimum energy 2 link-disjoint paths problem, as well as a polynomial time algorithm for the minimum energy k node-disjoint paths problem in [25].

6. CONCLUDING REMARKS

We showed applications of graph/network theory to several problems in communications and focused on the existence or non-existence of efficient algorithms to solve them. Since the demand for communication services continues to increase, various problems must be solved. Applications of old and new graph/network theory are also expected to provide insight to solve such problems with efficient algorithms or heuristic techniques. Finally, together with technical issues, a crucial social issue remains that must be settled before the use of various multi-hop wireless networks is allowed in public areas, because, apart from the fixed infrastructure of communications, no third watchers exist in the use of multi-hop wireless networks. The social issue is: what legal measures are required for permissible and reasonable use of multi-hop wireless networks in our societies?

7. ACKNOWLEDGEMENT

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