TRANSFORMATIONS OF HEEGAARD DIAGRAMS
BY THE ELEMENTARY DS-DEFORMATIONS II

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1. Introduction

Throughout this paper, we will be considering the piecewise linear standpoint of view.

[1] obtains transformations of Heegaard diagrams by using the elementary DS-deformations (H. Ikeda, M. Yamashita, & K. Yokoyama [6]) for DS-diagram (H. Ikeda & Y. Inoue [3]). The present paper shows that the two in these correspond to the well-known handle sliding and band move for Heegaard diagram (§ 5 Theorem 7). § 2 deals with the definitions concerning handlebodies, band move and handle sliding. § 3 deals with the definitions of Heegaard splittings, diagrams and Heegaard cut diagrams. § 4 states the handle sliding and band move for Heegaard diagram. § 5 states the definitions of elementary DS-deformations, and gives transformations of Heegaard cut diagrams by them. The theorem 7 is the main theme of this paper.

2. Handlebodies, handle sliding and band move

From now on, notation $\partial X$, $\text{Int}(X)$, $\text{Cl}(X)$ indicates the boundary, interior, closure of a point set $X$, respectively. $M^3$ denotes a closed, connected orientable 3-manifold unless otherwise stated.

Definition 1. Let $\{D_1, \ldots, D_n\}$ be mutually disjointed 2-disks and $h_i = D_i \sqcup [0,1] (i = 1, \ldots, n)$. A handlebody $H$ of genus $n (\sqcup 1)$ means a 3-ball (cube) $B^3$ with $n$ solid handles $\{h_i\}$ so that the result of attaching $h_i$ with homeomorphisms throws $2n$ disks $D_i \sqcup 0$, $D_i \sqcup 1$ onto $2n$ disjointed 2-disks on $\partial B^3$. $H$ is represented as $B^3 + \bigoplus_{i=1}^{n} (h_i)$ where $B^3 \sqcup h_i = \partial B^3 \sqcup h_i = (D_i \sqcup 0, D_i \sqcup 1)$.

We note that $\partial H$ is an orientable or nonorientable closed surface of Euler characteristic $2 - 2n$ according as $H$ is orientable or nonorientable.
**Definition 2.** Let $H$ be a genus $n$ handlebody and $\{D_i\}_{i=1, \ldots, n}$, mutually disjointed properly embedded 2-disks in $H$. If the $\partial(H - \{D_1 \sqcup \cdots \sqcup D_n\})$ becomes 3-ball, then the collection $\{D_i\}_{i=1, \ldots, n}$ is called a complete system of meridian disks of $H$ and $\{\partial D_i\}$ a complete system of meridian circles of $\partial H$. Note that $\{D_1, \ldots, D_n\}$ cuts $\partial H$ into 2-sphere with $2n$ holes.

**Definition 3.** In the following, let $H$ be an orientable genus $n$ handlebody with the same presentation as in the definition 1.

1. **Handle sliding**: let $h_i, h_j$ be a handle of $H$ shown in Fig. 1-1, respectively. By an ambient isotopy of $H$, keeping $D_i \sqcup 0$ fixed, and sliding $D_i \sqcup 1$ along the directions of the lines in $\partial(B^3 + h_i)$, as shown in Fig. 1-1, 1-2, $h_i$ goes over $h_j$ and turns back to the first place. This operation is called a handle sliding of $h_i$ about $h_j$.

![Fig. 1-1](image1)

![Fig. 1-2](image2)

2. **Band move**: let $\{D_i\}_{i=1, \ldots, n}$ be a complete system of meridian disks of $H$ and $\{m_i\} (m_i = \partial D_i)$ a complete system of meridian circle of $\partial H$. Let $\partial$ be an arc on $\partial H$ that joins two chosen meridians $m_i$, $m_j$ and $\text{Int}(\partial) \cap (m_i \cup m_j) = \phi$. See Fig. 1-3. Let $N(m_i + \partial + m_j, \partial H)$ be a regular neighborhood of $m_i + \partial + m_j$ on $\partial H$. $\partial N$ consist of three circles. Out of the three circles, two are isotopic to $m_i, m_j$ and then the remainder is not isotopic to them. Let the notation of remainder be $m_{ij}$. $m_{ij}$ is called a band sum of $m_i$ and $m_j$ (with respect to the band $\partial$). It has also the very pleasant property that bounds a disk and it is homeomorphic to $D_i$ and $D_j$. Changing the label $m_{ij}$ into $m_i (m_j \text{ resp})$ is called a band move of $m_i$ ($m_j \text{ resp}$).
There is an important result about the band move.

**Theorem 1** (Zieschang [7], Birman [8]). Let $H$ be a genus $n(\sqcup)2$ handlebody. Then any two complete systems of meridian circles of $\partial H$ transform each other under a finite sequence of band moves.

3. Heegaard splittings, diagrams and Heegaard cut diagrams

**Definition 4.** A closed, connected 3–manifold $M^3$ is represented with a union of two handlebodies $H_1$, $H_2$ along their boundaries in $M^3$: $M^3 = H_1 \sqcup H_2$ so that $H_1 \sqcup H_2 = \partial H_1 = \partial H_2$, $\partial H_1 = \partial H_2$ is a closed surface of genus $n$. Let the surface be $F$, $H_1(\sqcup H_2)$ resp. and $F$ are orientable or nonorientable according as $M^3$ is orientable or nonorientable. A triplet $(H_1, H_2, F)$ or $M^3 = H_1 \sqcup H_2$ is called a Heegaard splitting ($H$-splitting) with genus $n$ and $H_1(\sqcup H_2)$ resp. a Heegaard-handlebody ($H$-handlebody). $F$ is called a Heegaard-surface ($H$-surface) and the integer $n(\sqcup 0)$, Heegaard genus ($H$-genus). Let $U$ and $V$ be disjointed handlebodies with the same genus $n$. Let $f : U \sqcup V$ be a homeomorphism so that $f|\partial U \sqcup \partial V$ is an orientation-reversing homeomorphism. Gluing together $\partial U$ of $U$ and $\partial V$ of $V$ by $f$, we obtain $M^3$. Then $M^3$ is denoted as $(M^3; U, V, f)$ or $M^3 = U \sqcup V$. It is called a genus $n$ $H$-splitting of concerning $f$. In $(M^3; U, V, f)$, by replacing $f^{-1}(V)$ with $V$, one can regard $(M^3; U, V, f)$ as $(U, V, F)$ of $M^3$.

**Theorem 2.** Let $M^3 = H_1 \sqcup H_2$ and $M \sqcup M$ be two Heegaard splittings with the same genus. Suppose that there exist homeomorphisms $f : H_1 \sqcup H_1$ and $g : H_2 \sqcup H_2$ so that the right side diagram becomes commutative. Then $M^3$ is homeomorphic to $M \sqcup M$.

Proof. Suppose that $h : M^3 \sqcup M$ is a homeomorphism so that $h|\partial H_1 = f$ and $h|\partial H_2 = g$. Then by the commutative diagram, $h$ is well-defined. q.e.d.
Theorem 3. Let $M^3 = H_1 \sqcup H_2$ be a genus $n$ Heegaard splitting and $\Phi : H_1 \sqcup H_1$ a homeomorphism.

Let $M \Phi = H_1 \Phi \left( \partial (\partial H_1) \right)^{\perp} H_2$. Then $M \Phi$ is homeomorphic to $M^3$.

Proof. Let the elliptical character $id.$ be an identification map of $\partial H_2$. Then the right side diagram becomes commutative. Therefore, by the theorem 2, $M \Phi$ is homeomorphic to $M^3$. q.e.d.

By the above theorem, we can apply the handle sliding to H-splitting to examine the changing of $M^3$.

Next definition is Heegaard diagram.

Definition 5. Suppose $\langle H_1, H_2, F \rangle$ is a genus $n$ H-splitting of $M^3$. Let $\langle D_1, \emptyset, D_n \emptyset \rangle$, $\langle D_1 \emptyset, \emptyset, D_n \emptyset \rangle$ be a complete system of meridian disks of $U, V$ respectively, and $\{m\} = \{m_1, \emptyset, m_n\} = \{\emptyset D_1, \emptyset, \emptyset D_n \emptyset\}$.

Then $\langle H_1 : m, l \rangle (\langle H_2 : 1, m \rangle \text{ resp.})$ is called a genus $n$ Heegaard diagram (H-diagram) associated with $\langle H_1, H_2, F \rangle$. $\langle m, l \rangle (\langle l, m \rangle \text{ resp.})$ is called a meridian-longitude system of $\langle H_1 : m, l \rangle (\langle H_2 : 1, m \rangle \text{ resp.)}$.

Let $\langle H_1 : m_1, \emptyset, m_n, l_1, \emptyset, l_n \rangle$ be a genus $n$ H-diagram associated with $\langle H_1, H_2, F \rangle$. We may assume that $\langle m_1 \emptyset, \emptyset, m_n \rangle \emptyset (l_1 \emptyset \emptyset \emptyset l_n \emptyset)$ consists at most of finite points (by an argument of general position).

Definition 6. The number of finite points of $\langle m \rangle \emptyset (l) = \langle m_1 \emptyset \emptyset \emptyset m_n \rangle \emptyset (l_1 \emptyset \emptyset \emptyset l_n \emptyset)$ is called the number of the points of intersection with $\langle H_1 : m, l \rangle$ or $\langle H_2 : 1, m \rangle$.

Next we define the Heegaard cut diagram from a Heegaard diagram.

Let $\langle U, V, F \rangle$ be a genus $n (\emptyset 1)$ H-splitting of $M^3$ and $\{D_i \emptyset \text{ resp.} \}$ ($i = 1, \emptyset, n$) a complete system of meridian disks of $U(V \text{ resp.})$. Let $\langle U : m, l \rangle (\langle V : 1, m \rangle \text{ resp.)}$ be a genus $n$ H-diagram of $\langle U, V, F \rangle$. $\{m\} = \{m_1, \emptyset, m_n\} = \{\emptyset D_1, \emptyset, \emptyset D_n \emptyset\}$ and $\{l\} = \{l_1, \emptyset, l_n\} = \{\emptyset D_1 \emptyset \emptyset, \emptyset D_n \emptyset\}$. Suppose each circle $l_i, m_i$ is oriented. By cutting off $M^3$ at $F$, disjointed genus $n$ handlebodies $U, V$ put on the former labels are obtained. We put the same label $F$ on $\partial U, \partial V$, 

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respectively. Let \( \{ j_1, j_2, \ldots, j_{l_j} \} \) be the ordering points of intersection of \( l_j \cap (m_1 \cap m_2 \cap \cdots \cap m_n) \) in \( \partial \mathcal{U} \). \((m_1, \ldots, m_n) \) in \( \partial \mathcal{U} \) decomposes each \( I_j \) into edges at those points. We put labels \( j_1(l_j) j_2(l_j) j_3(l_j), \cdots, j_{l_j}(l_j) j_1 \), in these orders, to these edges in \( I_j \) according to the orientation of \( I_j \) such as \( I_j = j_1(l_j) j_2(j_2(l_j) j_3 \cdots j_{l_j}(l_j) j_1 \). We may assume that each label \( j_k(l_j) j_{k+1} \) is oriented with the same orientation as \( I_j \). The inverse orientation of \( j_k(l_j) j_{k+1} \) is denoted by \((j_k(l_j) j_{k+1})^{-1} \) or \((j_{k+1}(l_j^{-1}) j_k) \). Conversely \((I_1, \ldots, I_n) \) in \( \partial \mathcal{U} \) decomposes each \( m_i \) into edges such as \( m_i = i_1(m_i) i_2 \cdots i_3 \cdots i_{l_i}(m_i) i_1 \). By cutting off \( U \) at each \( D_i \), a 3-ball \( B_i^3 \) is obtained. \( \partial B_i^3 \) is a 2-sphere \( S_i^2 \) and \( n \) pairs of disks \( D_i^+ j_i \), \( D_i^- j_i \) appear in \( S_i^2 \). Since both \( \partial D_i^+ \) and \( \partial D_i^- \) are decomposed by the same edges in \( m_i \), they have oriented labels \( i_1(m_i) i_2 \cdots i_3 \cdots i_{l_i}(m_i) i_1 \) in common. Then we have a planar 3-regular graph described as a diagram over a plane \( (N \cap \partial \mathcal{U}) \) if a point \( \partial \cap \partial S_i^2 \) is designated.

Therefore, we now go to the next definition.

**Definition 7.** A planar 3-regular graph

\[
\{(m_i \cap \partial D_i^+ = i_1(m_i) i_2 \cdots i_3 \cdots i_{l_i}(m_i) i_1, (m_i \cap \partial D_i^- = i_1(m_i) i_2 \cdots i_3 \cdots i_{l_i}(m_i) i_1, j_1(l_j) j_2(j_2(l_j) j_3 \cdots j_{l_j}(l_j) j_1), \cdots, j_{l_j}(l_j) j_1) \} \quad (i, j = 1, \ldots, n)
\]

is called a Heegaard cut diagram (H-cut-diagram) associated with \( \langle U : m, l \rangle \) and is described as \( \text{HCD}(m,l) \). In like manner \( \text{HCD}(l,m) = \langle V : l, m \rangle \) is defined and its expression is

\[
\text{HCD}(l,m) = \{(l_j - \partial D_j = j_1(l_j) j_2(j_2(l_j) j_3 \cdots j_{l_j}(l_j) j_1, (l_j = \partial D_j = j_1(l_j) j_2(j_2(l_j) j_3 \cdots j_{l_j}(l_j) j_1, i_1(m_i) i_2 \cdots i_3 \cdots i_{l_i}(m_i) i_1) \} \quad (i, j = 1, \ldots, n).
\]

Note that the set of vertices \( \frac{k}{l_j} (I_1, I_2, I_3, \ldots, I_l_j) \) equals to \( \frac{k}{l_j} (l_j, j_2, j_3, \ldots, j_l_j) \).

**Remark 1.** If the handlebody \( U \) of \( \langle U : m, l \rangle \) is represented such as \( B^3 j \cap \mathcal{P} (h_i) \) in Def. 1, then there exists the H-cut-diagram \( \text{HCD}(m,l) \) of \( \langle U : m, l \rangle \) in \( \partial B_i^3 \).

Let \( |\text{HCD}(m,l)| \) be the presentation of the underlying space of labels of \( \text{HCD}(m,l) \). Let \( \partial \), \( \partial \cup \partial \) be faces that are the connected components of \( \text{Cl}(S_i^2 - |\text{HCD}(m,l)| = \frac{k}{l_j} (D_j^+, D_j^-) \).

Let the name \( \partial \) of face be its label. Since \( \partial \mathcal{U} \) and \( \partial \mathcal{V} \) are put on the same label \( F \), there exists the face that should be put on the same label \( \partial \) in \( |\text{HCD}(l,m)|, \{m\} \cup \{l\} \) consists of points of intersection of \( \langle U : m, l \rangle \) or \( \langle V : l, m \rangle \). Since there exist the same labeled two edges \( i_k(m_i) i_{k+1} (\partial m_i = \partial D_i^+), i_k(m_i) i_{k+1} (\partial m_i = \partial D_i^-) \) in \( \text{HCD}(m,l) \), there exist the same labeled two points \( i_k \) in \( \text{HCD}(m,l) \) (see 5 Fig. (CU1-A) \( \cap \) (CV1-A)). Since there is \( i_k(m_i) i_{k+1} \) in \( \text{HCD}(l,m) \),

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there exist $D_j \subseteq D_j$ such as $i_k \subseteq D_j \subseteq$ and $i_k \subseteq D_j \subseteq$ in $|HCD(l,m)|$. We put the same label $D_i \subseteq D_j \subseteq$ (resp.) on the two disks $D_i \subseteq D_j \subseteq$ (resp.) in $|HCD(m,l)|$, $|HCD(l,m)|$ resp.). Therefore by counting the same labels of the vertices, edges and faces in $|HCD(m,l)|$, we have:

**Proposition 4.** There exist the same labeled four vertices, the same oriented labeled three edges and the same oriented labeled two faces, i.e., 2-disks or punctured 2-disks (= disk with $n(\sqcup 1)$ holes) in $|HCD(m,l)|$. $|HCD(l,m)|$.

4. Transformations of Heegaard diagrams by the handle slidings and band moves

In this section, we give basic transformations of Heegaard diagrams.

Let $(U,V,F)$ be a Heegaard splitting of $\mathcal{M}^3$. Let $(U:m,l)\sqcup (V:l,m)$ be Heegaard diagrams associated with $(U,V,F)$, $(m,l)\sqcup (l,m)$ resp.) is a meridian-longitude system of $(U:m,l)$ $(V:l,m)$ resp.). Let the following figure U1-A be a part of $(U:m,l)$. At this time, generally, longitudes are running as follows to two handles $h_i$ and $h_j$. The longitudes $\{l_{ij}, i_{ij}\}$ that are drawn heavily go around side by side on the two handles $h_i$ and $h_j$. The longitudes $\{l_{ij}, i_{ij}h_i\}$ go around on $h_i$, respectively. Let V1-A be a part of $(V:l,m)$. V1-A is the dual part of U1-A. The longitudes $m_i, m_j$ drawn heavily crosses the meridians $\{l_{ij}, i_{ij}h_i\}$, $\{l_{ij}, i_{ij}h_j\}$, respectively. In U1-B, longitudes $\{l_{ij}, i_{ij}\}$ go around on $h_i$ (not on $h_j$), $\{l_{ij}, i_{ij}\}$ go around on both the $h_i$ and $h_j$, and $\{l_{ij}, i_{ij}\}$ go around on $h_j$. V1-B is the dual part of U1-B. The longitude $m_i$ crosses meridians $\{l_{ij}, i_{ij}h_i\}$, and $m_j$ crosses $\{l_{ij}, i_{ij}h_j\}$.

![Fig. U1\(\text{A}\)](-84-)


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Then we have;

**Theorem 5.** There exist transformations from \((U_1-A) \sqcup (V_1-A)\) into \((U_1-B) \sqcup (V_1-B)\) of \((U;m,l) \sqcup (V;l,m)\). The transformation from \(U_1-A\) into \(U_1-B(V_1-A\) into \(V_1-B\) resp.) do not change the Heegaard genus, but change the number of the points of intersection as many as \(|I - p|\).

Proof. By the handle sliding of the leg of \(h_i\) that is nearer \(h_j\) to \(h_j\), \(U_1-B\) is obtained. Corresponding to this transformation, \(V_1-B\) is obtained from \(V_1-A\). This transformation shows a band sum and a band move of \(m_j\) in the place where \(m_i\) and \(m_j\) on \(\partial V\) are running across the same meridians \(\{l_{ij}, \sqcup, l_{ji}\}\). q.e.d.

In like manners, the transformation of \(U_1-A\) which corresponds to a handle sliding of \(h_j\) about \(h_i\), and the band move of \(m_i\) on \(\partial V\) are obtained.

5. Transformations of Heegaard diagrams by the elementary DS-deformations

We begin with the definition of the elementary DS-deformations for DS-diagrams.

**Definition 8.** The following figure \(G-i\) \((i = 1, 2, 3, 4)\) is a part of DS-diagrams, respectively. There exist two types of the elementary DS-deformations for DS-diagrams:

1. The deformation from \(G-1\) to \(G-2(G-1 \sqcup G-2)\) is called \(D_3^+\)-deformation and \(G-2 \sqcup G-1\) is called \(D_3^−\)-deformation. They are generally called \(D_3\)-deformations.
2. \(G-3 \sqcup G-4\) is called \(D_2^+\)-deformation and \(G-4 \sqcup G-3\) is called \(D_2^−\)-deformation. They are generally called \(D_2\)-deformations.

The definitions of (1) and (2) are generally called the elementary DS-deformations for DS-diagrams. They hold the conditions of Proposition 4. The figure \(P-i\) \((i = 1, 2, 3, 4)\) is the polyhedron that corresponds to \(G-i\), respectively.
It is known that the elementary $DS$-deformations preserve homeomorphism of a 3-manifold. According to the Proposition 4, we see that $HCD(m,l) \sqcup HCD(l,m)$ has the same structure of a $DS$-diagram (Yamashita [5]) without any connectedness (components of a $DS$-diagram are one and those of $HCD(m,l) \sqcup HCD(l,m)$, more than one). Therefore we have;

**Theorem 6.** The $DS$-deformations ([6]) for $DS$-diagram are applied to Heegaard cut diagrams $HCD(m,l) \sqcup HCD(l,m)$.

From now on, we carry out transformations of theorem 5 with a finite sequence of
elementary DS-deformations. Let the figure CU1-A, CV1-A be the part of $HCD(m, l)$, $HCD(l, m)$ that corresponds to U1-A, V1-A, respectively. For this transformation, we put label to each edge in $\langle CU1-A \rangle \sqcup \langle CV1-A \rangle$ according to the rule of Proposition 4, i.e., longitudes and meridians in U1-A(V1-A resp.) decompose each other into labeled edges as follows: Here we omit the labels of vertexes of edges.

\[
l_{i0} = \square A_0 \cdot B_0 \cdot C_0 \square (\square = 1, \square, l), \quad l_{i0} = \square S_0 \cdot T_0 \square (\square = 1, \square, p),
\]

\[
l_{j0} = \square V_0 \cdot W_0 \square (\square = 1, \square, q), \quad m_i = m_{i1} \cdot m_{i2} \sqcup m_{i3} \sqcup m_{i4}, \quad m_j = m_{j1} \cdot m_{j2} \sqcup m_{j3} \sqcup m_{j4}
\]

Let the figure PU1-A, PV1-A be the polyhedron that corresponds to CU1-A, CV1-A, respectively.

Step 1. We apply $D_3^+$-deformation to $\langle CU1-A \rangle \sqcup \langle CV1-A \rangle$ shown as the dotted line on its. Then we obtain $\langle CU1-A1 \rangle \sqcup \langle CV1-A1 \rangle$. We denote these processes by symbols as follows:

\[
D_3^+\langle \langle CU1-A \rangle \sqcup \langle CV1-A \rangle \rangle \Rightarrow \langle \langle CU1-A1 \rangle \sqcup \langle CV1-A1 \rangle \rangle
\]

Corresponding to those transformations, the polyhedron PU1-A(PV1-A resp.) transforms into PU1-A1(PV1-A1 resp.) by the deformation from P-1 to P-2. We denote these processes by symbols as follows:

\[
D_3^+\langle \langle PU1-A \rangle \sqcup \langle PV1-A \rangle \rangle \Rightarrow \langle \langle PU1-A1 \rangle \sqcup \langle PV1-A1 \rangle \rangle
\]

Hereafter we will use the above notations suitably.

**Remark 2.** Neither CU1-A1 nor CV1-A1 is H-cut-diagram but $\langle CU1-A1 \rangle \sqcup \langle CV1-A1 \rangle$ give a presentation of $M^3$ from a viewpoint of DS-diagram.
CU1 A1

CV1 A1

PU1 A1
We continue elementary $DS$-deformations.

Step $i (i = 2, \ldots , \ell)$. We carry out $D_{i-1}^-$-deformation continuously from step 2 to step $l$ as follows:

\[ D_{i-1}^- ((CU_1-Ai - 1) \sqcup (CV_1-Ai - 1)) \Rightarrow (CU_1-Ai) \sqcup (CV_1-Ai) \]

\[ D_{i-1}^- ((PU_1-Ai - 1) \sqcup (PV_1-Ai - 1)) \Rightarrow (PU_1-Ai) \sqcup (PV_1-Ai) \]

After this, there are figures $(CU_1-A2) \sqcup (CV_1-A2), P-2_1-A1, P-1_1-A2, (PU_1-A2) \sqcup (PV_1-A2), (CU_1-A1 - 1) \sqcup (CV_1-A1 - 1), (PU_1-A1 - 1) \sqcup (PV_1-A1 - 1), (CU_1-A1) \sqcup (CV_1-A1), (PU_1-A1) \sqcup (PV_1-A1)$. These are figures in the part of the above-mentioned deformations.
The following figures $P-2\_1-A1$ and $P-1\_1-A2$ show the $D_3$-deformation from the polyhedron PU1-A1 to PU1-A2.
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PV1[A] - 1

CU1[A]

CV1[A]
The circular cylinder which is shown by the dotted lines in Fig. PV1-Al shows $D_3^+$-deformation.
The right figure pictures PU1–Al above once more in another point of view.

Now we divide in case of \( p = 0 \) and \( p \not\equiv 1 \).

Step \( l + 1 \). If \( p = 0 \), we apply \( D_2^- \)-deformation to \( (CU1–Al) \# (CV1–Al) \), then we get \( H \)-cut–diagrams that longitudes do the way of running which are the same as \( H \)-cut–diagrams \( HCD(m, l) \# HCD(l, m) \) of \( (U1–B) \# (V1–B) \).

Remark 3. If \( p = 0 \), then there exists a wave in the part of CU1–A (to know the definition, for instance, see Homma, Ochiai, Takahashi [9]).

Step \( l + i (i = 1, \not\equiv ., p) \). If \( p \not\equiv 1 \), then we carry out \( D_3^+ \)-deformation continuously from step \( l + 1 \) to step \( l + p \) as follows:

\[
D_3^+((CU1–Al + i - 1) \# (CV1–Al + i - 1)) \Rightarrow (CU1–Al + i) \# (CV1–Al + i)
\]

\[
D_3^+((PU1–Al + i - 1) \# (PV1–Al + i - 1)) \Rightarrow (PU1–Al + i) \# (PV1–Al + i)
\]

Step \( l + p + 1 \). \( D_2^-((CU1–Al + p) \# (CV1–Al + p)) \Rightarrow (CU1–Al + p + 1) \# (CV1–Al + p + 1)
\]

\[
D_2^-((PU1–Al + p) \# (PV1–Al + p)) \Rightarrow (PU1–Al + p + 1) \# (PV1–Al + p + 1)
\]

The above \( D_2^- \)-deformation connects two edges \( m_{ij}f_p \) resp. and \( g_p(n_1) \) resp. Therefore, we put two labels \( g_pm_{ij} | n_1f_p \) resp. to this connected edge. The ways of running in longitudes of \( (CU1–Al + p + 1) \# (CV1–Al + p + 1) \) are the same as of \( HCD(m, l) \# HCD(l, m) \) of \( (U1–B) \# (V1–B) \).

Therefore we have:

**Theorem 7.** A finite sequence of elementary DS-deformations that erases a meridian of Heegaard (-cut-) diagram corresponds to the handle sliding and band move of Heegaard diagrams.

After this, there are figures \( (CU1–Al + 1) \# (CV1–Al + 1), (PU1–Al + 1) \# (PV1–Al + 1), (CU1–Al + 2) \# (CV1–Al + 2), (PU1–Al + 2) \# (PV1–Al + 2), (CU1–Al + p) \# (CV1–Al + p), (PU1–Al + p) \# (PV1–Al + p), P-4_1–Al + p, P-3_1–Al + p + 1, (CU1–Al + p + 1) \# (CV1–Al + p + 1) \# (V1–B) \).
1), (PU1−AI + p + 1) ∩ (PV1−AI + p + 1). These are figures in the part of the above-mentioned deformations.
PU1(\mathbb{A}^I_\mathbb{A}) + \beta + 1

PV1(\mathbb{A}^I_\mathbb{A}) + \beta + 1

It is so difficult for us to carry out the handle sliding and band move for H-diagrams. Because, these transformations are done on the H-surfaces. Moreover, it is also difficult for a computer to transform these. However, if we take much time, by the theorem 7, we can handle the handle sliding and band move by the DS-deformations on a plane. Moreover, these transformations are an algorithm. Therefore, a computer can transforms the H(-cut)-
diagrams.

References


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