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# Initial Boundary Value Problem for the Spherically Symmetric Motion of Isentropic Gas

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We study the spherically symmetric motion of an ideal gas surrounding a solid ball. This is governed by the compressible Euler equation of isentropic gas dynamics. The associated initial boundary value problem is solved by using the compensated compactness method for initial data containing the vacuum. The constructed weak solutions are temporally local but the class of initial data includes the stationary solutions.

*Key words*: initial boundary value problem, isentropic gas dynamics, spherically symmetric motion, weak solutions

Let us consider the system of equations

$$(1)\qquad \begin{aligned} &\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{2}{r}\rho u = 0 \\ &\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) + \frac{\partial p}{\partial r} = -\rho\frac{M}{r^2} \end{aligned}$$

$$(2)\qquad p = A\rho^{\gamma}$$

on $t \geqq 0$ and $1 \leqq r < +\infty$. Here $M$, $A$ and $\gamma$ are positive constants such that $1 < \gamma \leqq 5/3$. This system governs the isentropic and spherically symmetric motion of an atmosphere surrounding a solid star with radius 1 and mass $M$. The variable $\rho$ means the density, $u$ the velocity, $p$ the pressure and $-M/r^2$ stands for the gravitational force. The problem is to find a solution of the system (1) (2) which satisfies the initial condition

$$(3)\qquad \rho\,|_{t=0} = \rho^0(r), \quad u\,|_{t=0} = u^0(r)$$

and the boundary condition

$$(4)\qquad \rho u\,|_{r=1} = 0.$$

We are interested in the case where the support of $\rho^0(r)$ is compact in $[1, +\infty)$. In the article [4] we established the local existence of "tame" solutions. However this class of "tame" solutions does not include the stationary solutions

$$\rho = \begin{cases} \left[\dfrac{(\gamma-1)M}{\gamma A}\right]^{1/(\gamma-1)} \left(\dfrac{1}{r} - \dfrac{1}{R}\right)^{1/(\gamma-1)} & (1 \leqq r \leqq R) \\ 0 & (R < r) \end{cases}$$

$$u = 0,$$

where $R$ is an arbitrary constant such that $R > 1$. In this paper we shall try to establish the existence of weak solutions in order to surmount the above limitation of [4].

First we rewrite the system (1) in the form as a perturbation of a system of conservation laws as follows:

$$(5) \qquad \begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u) = h_1 = -\frac{2}{r}\rho u \\ &\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial r}(\rho u^2 + p) = h_2 = -\frac{2}{r}\rho u^2 - \rho\frac{M}{r^2}. \end{aligned}$$

The system $(5)_0$ obtained by substituting 0, 0 to $h_1, h_2$ in (5) is the system governing the one dimensional motion of an isentropic gas. Initial boundary value problems associated with $(5)_0$ were studied by S. Takano [7]. On the other hand the initial value problem without boundary conditions for non-homogeneous systems ($h \neq 0$) was studied by X. Ding et al [3]. Therefore one can expect to solve our problem by combining these two studies. This is our way of investigation.

Introducing the vector variable $U = {}^t(\rho, m)$, where $m = \rho u$, we can write (5) as

$$(5)' \qquad U_t + f(U)_r = H(r, U),$$

where $f(U) = {}^t(f_1, f_2)$ with $f_1 = m$, $f_2 = \frac{m^2}{\rho} + p$, $H(r, U) = {}^t(H_1, H_2)$ with $H_1 = -\frac{2m}{r}$, and $H_2 = -\frac{2m^2}{r\rho} - \rho\frac{M}{r^2}$. Since we consider solutions such that $0 \leqq \rho \leqq C$ and $|m| \leqq C\rho$, we interpret $m^2/\rho$ as 0 when $\rho = m = 0$. The associated homogeneous system is

$$(5)_0{}' \qquad U_t + f(U)_x = 0.$$

The eigenvalues of the matrix $Df(U)$ are $\lambda_1 = u - c$, $\lambda_2 = u + c$, where $c = \sqrt{dp/d\rho} = \sqrt{A\gamma}\rho^{(\gamma-1)/2}$. The Riemann invariants are

$$(6) \qquad w_1 = u + \frac{2}{\gamma - 1}c, \quad w_2 = u - \frac{2}{\gamma - 1}c.$$

Given $B > 0$, we define the domain $\Sigma(B)$ by

$$\Sigma(B) = \{(\rho, u) \mid 0 \leqq \rho,\ w_1 \leqq B,\ -w_2 \leqq B\}.$$

The Riemann problem of $(5)_0'$ is the Cauchy problem for the initial condition

$$U\mid_{t=0}=\begin{cases} U_L & (x<0) \\ U_R & (x>0). \end{cases} \tag{7}$$

This problem is solved uniquely in a standard manner. Hereafter we shall denote this solution of $(5)_0'$ (7) by $U = U_0(x,t;U_L,U_R)$. The domain $\Sigma(B)$ is invariant, that is, if $U_L$ and $U_R$ belong to $\Sigma(B)$, then the solution $U_0(t,x;U_L,U_R)$ remains in $\Sigma(B)$. Moreover if $U(q)\in\Sigma(B)$ for $a\leqq q\leqq b$ then the mean value $\int_a^b U(q)dq/(b-a)$ belongs to $\Sigma(B)$.

A weak solution of (5) (4) (3) is a field $U\in L^\infty([1,+\infty)\times[0,T))$ such that $0\leqq\rho\leqq C$, $|m|\leqq C\rho$ which satisfies

$$\int_0^T\int_1^\infty (U\cdot\Phi_t+f(U)\cdot\Phi_r+H(r,U)\cdot\Phi)dr\,dt + \int_1^\infty \Phi\mid_{t=0} U^0 dr = 0$$

for any test function $\Phi={}^t(\phi_1,\phi_2)\in C_0^\infty([1,+\infty)\times[0,T))$ such that $\phi_2\mid_{r=1}=0$. Here of course $U^0=(\rho^0,\rho^0u^0)$.

Let us construct approximate solutions by the Lax-Friedrichs scheme. Suppose that $\rho^0$ and $u^0$ are bounded, $\rho^0\geqq 0$ and $\rho^0$ has a compact support. Then we can find a sufficiently large $B^0$ such that $(\rho^0(r),u^0(r))\in\Sigma(B^0)$ for $r\geqq 1$. Fix $B^{00}$ such that $B^0<B^{00}$ and $T$ such that

$$\sqrt{\frac{(\gamma-1)M}{4}}T<\operatorname{Arctan}\sqrt{\frac{\gamma-1}{4M}}B^{00}-\operatorname{Arctan}\sqrt{\frac{\gamma-1}{4M}}B^0.$$

We denote by $\Delta r=\Delta$ and $\Delta t$ the mesh lengths for $r$ and $t$ which satisfy $\Delta r/\Delta t > 2\Lambda_0$, the Courant-Friedrichs-Lewy condition, where $\Lambda_0>\sup\{|\lambda_1|,|\lambda_2| \mid U\in\Sigma(B^{00})\}$.

Suppose that the approximate solution $U^\Delta(r,t)$ has been defined for $1\leqq r<+\infty$, $0\leqq t<(n-1)\Delta t$ such that $\rho^\Delta\geqq 0$ and $B_\nu=\sup\{w_1^\Delta(r,t),-w_2^\Delta(r,t)\mid 1\leqq r<+\infty,\ (\nu-1)\Delta t\leqq t<\nu\Delta t\}<B^{00}$ for $\nu=1,2,\dots,n-1$. We put

$$U_{n-1,j}^\Delta=\frac{1}{2\Delta}\int_{1+(j-1)\Delta}^{1+(j+1)\Delta}U^\Delta(r,(n-1)\Delta t-0)dr$$

for $j=1,2,\dots$. But when $n$ is even, we put

$$U_{n-1,1}^\Delta=U_{n-1,2}^\Delta=\frac{1}{3\Delta}\int_1^{1+3\Delta}U^\Delta(r,(n-1)\Delta t-0)dr.$$

Of course

$$U_{0,j}^\Delta=\frac{1}{2\Delta}\int_{1+(j-1)\Delta}^{1+(j+1)\Delta}U^0(r)dr.$$

If $\rho^{\Delta}_{n-1,j}=0$, then we interpret

$$u^{\Delta}_{n-1,j}=\frac{1}{2\Delta}\int_{1+(j-1)\Delta}^{1+(j+1)\Delta}u^{\Delta}(r,(n-1)\Delta t-0)dr.$$

We are going to define $U^{\Delta}_0$ for $(n-1)\Delta t \leqq t < n\Delta t$. Given $k \geqq 2$ such that $n+k$ is odd, we put

$$U^{\Delta}_0(r,t)=U_0(r-(1+k\Delta),t-(n-1)\Delta t;\ U^{\Delta}_{n-1,k-1},U^{\Delta}_{n-1,k+1})$$

for $1+(k-1)\Delta \leqq r < 1+(k+1)\Delta$ and $(n-1)\Delta t \leqq t < n\Delta t$. For $1 \leqq r < 1+\Delta$ and $(n-1)\Delta t \leqq t < n\Delta t$, we define

$$U^{\Delta}_0(r,t)=U_0(r-1,t-(n-1)\Delta t;\ U^{\Delta}_{n-1,0},U^{\Delta}_{n-1,1}),$$

where $U^{\Delta}_{n-1,0}$ is defined according to the boundary condition (4) as follows. 1) If $U^{\Delta}_{n-1,1}$ satisfies $\rho>0$, $u \leqq 0$, there exists $U^{\Delta}_{n-1,0}$ with $u=0$ from which $U^{\Delta}_{n-1,1}$ is connected by a 2-shock with positive velocity; 2) if $U^{\Delta}_{n-1,1}$ satisfies $u \geqq 0$, $w_2 \leqq 0$, then there exists $U^{\Delta}_{n-1,0}$ with $u=0$ from which $U^{\Delta}_{n-1,1}$ is connected by a 2-rarefaction; 3) if $U^{\Delta}_{n-1,1}$ satisfies $u \geqq 0$, $w_2 \geqq 0$, then there exists $U^*$ with $\rho=0$ from which $U^{\Delta}_{n-1,1}$ is connected by a 2-rarefaction, and $U^*$ and $U^{\Delta}_{n-1,0}:\rho=u=0$ are connected by the vacuum; 4) if $U^{\Delta}_{n-1,1}$ satisfies $u \leqq 0$, $\rho=0$, then $U^{\Delta}_{n-1,0}:\rho=u=0$ is connected from $U^{\Delta}_{n-1,1}$ by the vacuum. If $n$ is even, we put

$$U^{\Delta}_0(r,t)=U^{\Delta}_{n-1,1}=U^{\Delta}_{n-1,2}$$

for $1+\Delta \leqq r < 1+2\Delta$.

Next, let us define, for $(n-1)\Delta t \leqq t < n\Delta t$,

$$\rho^{\Delta}(r,t)=\rho^{\Delta}_0(r,t)-\frac{2}{r}\rho^{\Delta}_0(r,t)u^{\Delta}_0(r,t)(t-(n-1)\Delta t),$$

$$u^{\Delta}(r,t)=u^{\Delta}_0(r,t)-\frac{M}{r^2}(t-(n-1)\Delta t).$$

(If $\Delta \leqq \Lambda_0/B_{n-1}$, then $1-\frac{2}{r}u^{\Delta}_0(t-(n-1)\Delta t)>0$, from which $\rho^{\Delta}(r,t)\geqq 0$ for $(n-1)\Delta t \leqq t < n\Delta t$.) Then we have

$$U^{\Delta}(r,t)=U^{\Delta}_0(r,t)+\widetilde{H}(r,U^{\Delta}_0(r,t),t-(n-1)\Delta t)(t-(n-1)\Delta t),$$

where $\widetilde{H}(r,U,s)={}^t(H_1(r,U),\ \widetilde{H}_2(r,U,s))$ with $\widetilde{H}_2(r,U,s)=H_2(r,U)+\frac{2M}{r^3}ms$. On the other hand we have

$$c^{\Delta}=\sqrt{A\gamma}(\rho^{\Delta})^{(\gamma-1)/2}=c^{\Delta}_0+g(r,c^{\Delta}_0,u^{\Delta}_0,t-(n-1)\Delta t)(t-(n-1)\Delta t),$$

where

$$g(r,c,u,s)=\begin{cases} c((1-\frac{2}{r}us)^{(\gamma-1)/2}-1)/s & (s>0)\\ -c\frac{\gamma-1}{r}u & (s=0).\end{cases}$$

We see that $g \leqq \frac{(\gamma-1)^2}{8}B^2$ provided that $(c,u)\in\Sigma(B)$, $r\geqq 1$, and $s\geqq 0$. Thus we have

$$w_1^\Delta = w_{01}^\Delta + G_1(r,w_0^\Delta,t-(n-1)\Delta t)(t-(n-1)\Delta t),$$
$$w_2^\Delta = w_{02}^\Delta + G_2(r,w_0^\Delta,t-(n-1)\Delta t)(t-(n-1)\Delta t),$$

where

$$G_1(r,w,s) = -\frac{M}{r^2}+\frac{2}{\gamma-1}g(r,c,u,s),$$
$$G_2(r,w,s) = -\frac{M}{r^2}-\frac{2}{\gamma-1}g(r,c,u,s).$$

Putting $B_n=\sup\{w_1^\Delta, -w_2^\Delta \mid 1\leqq r,\ (n-1)\Delta t\leqq t< n\Delta t\}$, we have

$$B_n \leqq B_{n-1}+\left(M+\frac{\gamma-1}{4}{B_{n-1}}^2\right)\Delta t.$$

Comparing this inequality with the ordinary differential equation

$$\frac{d\beta}{dt}=M+\frac{\gamma-1}{4}\beta^2,\quad \beta(0)=B^0,$$

whose solution is

$$\beta(t)=\sqrt{\frac{4M}{\gamma-1}}\tan\left(\sqrt{\frac{(\gamma-1)M}{4}}t+\mathrm{Arctan}\sqrt{\frac{\gamma-1}{4M}}B^0\right),$$

we have the estimate $B_n\leqq\beta(n\Delta t)$. Since $n\Delta t<T$, we have $B_n<B^{00}$ and we can continue constructing the approximate solution.

An entropy pair $(\eta,q)$ is a couple of functions $\eta,q\in C^1(\Sigma(B^{00}))\cap C^2(\Sigma(B^{00})\setminus(\rho=0))$ such that $Dq=D\eta\cdot Df$, where $D=(\partial/\partial\rho,\partial/\partial m)$ and $\eta|_{\rho=0}=0$. The kinetic energy

$$\eta^*=\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}p$$

and its flux

$$q^*=\left(\frac{1}{2}\rho u^2+\frac{\gamma}{\gamma-1}p\right)u$$

is an entropy pair. Then we have the following

Proposition 1. *Let $(\eta,q)$ be an entropy pair and $\Omega$ be a bounded open subset of $[1,+\infty)\times[0,T)$. Then $\{\eta(U^\Delta(r,t))_t+q(U^\Delta(r,t))_r\}_{0\leqq\Delta\leqq\Delta_0}$ is relatively compact in $H^{-1}_{\mathrm{loc}}(\Omega)$.*

*Proof.* The proof is almost the same as that of [3], Theorem 5. Suppose that $\mathrm{Supp}\, U \in [1, R) \times [0, T]$ $(R > \Lambda_0 T + R(0))$. Let $\phi$ be a test function on $\Omega$. Then we can write

$$\int_0^T \int_1^R (\eta(U^\Delta)\phi_t + q(U^\Delta)\phi_r) dr\, dt = (L_0 + L_1 + L_2 + L_3 + L_4)\phi, \tag{8}$$

where

$$L_0\phi = \int_0^T \int_1^R ((\eta(U^\Delta) - \eta(U_0^\Delta))\phi_t + (q(U^\Delta) - q(U_0^\Delta))\phi_r) dr\, dt,$$

$$L_1\phi = \int_1^R \phi(r, T)\eta(U_0^\Delta(r, T)) dr - \int_1^R \phi(r, 0)\eta(U_0^\Delta(r, 0)) dr,$$

$$L_2\phi = \sum_n \int_1^R \phi(r, n\Delta t)(\eta(U_0^\Delta(r, n\Delta t - 0)) - \eta(U_0^\Delta(r, n\Delta t + 0))) dr,$$

$$L_3\phi = \int_0^T \sum_{\text{shock}} (\sigma[\eta]_0 - [q]_0)\phi dt,$$

$$L_4\phi = -\int_0^T q(U_0^\Delta(1, t))\phi(1, t) dt.$$

Moreover we put $L_2 = L_{21} + L_{22} + L_{23}$ with

$$L_{21}\phi = \sum_{n,k} \phi_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} (\eta(U^\Delta(r, n\Delta t - 0)) - \eta(U_0^\Delta(r, n\Delta t + 0))) dr,$$

$$L_{22}\phi = \sum_n \int_1^R (\eta(U_0^\Delta(r, n\Delta t - 0)) - \eta(U^\Delta(r, n\Delta t - 0)))\phi(r, n\Delta t) dr,$$

$$L_{23}\phi = \sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} (\eta(U^\Delta(r, n\Delta t - 0)) - \eta(U_0^\Delta(r, n\Delta t + 0)))(\phi(r, n\Delta t) - \phi_{n,k}) dr,$$

where $\phi_{n,k} = \phi(1 + k\Delta, n\Delta t)$. The summation is taken over $n$ and $k$ such that $n + k$ is odd. (When $n$ is odd and $k = 2$, then $1 + (k - 1)\Delta$ stands for 1.)

Substituting $\eta = \eta^*$, $q = q^*$, $\phi = 1$, we get

$$\sum_n \int_1^R (\eta^*(U_0^\Delta(r, n\Delta t - 0)) - \eta^*(U_0^\Delta(r, n\Delta t + 0))) dr + \int_0^T \sum_{\text{shock}} (\sigma[\eta^*]_0 - [q^*]_0) dt \leqq C,$$

while

$$\sum_n \int_1^R (\eta^*(U_0^\Delta(r, n\Delta t - 0) - \eta^*(U_0^\Delta(r, n\Delta t + 0)))dr$$

$$= \sum_n \int_1^R \int_0^1 (1-\theta)^t (U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)) \cdot D^2\eta^*(U_0^\Delta r, n\Delta t + 0)$$
$$+ \theta(U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)))$$
$$\cdot (U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0))d\theta\, dr$$
$$- \sum_n \int_1^R \int_0^1 D\eta^*(U_0^\Delta(r, n\Delta t - 0) + \theta(U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t - 0)))$$
$$\cdot (U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t - 0))d\theta\, dr.$$

But the second term of the right hand side is estimated by

$$\sum_n \int_1^R \int_0^1 |D\eta^*| d\theta |\widetilde{H}| \Delta t\, dr \leqq C.$$

Since $\sigma[\eta^*]_0 - [q^*]_0 \geqq 0$, we have

$$\int_0^T \sum_{\text{shock}} (\sigma[\eta^*]_0 - [q^*]_0)dt \leqq C,$$

$$\sum_n \int_1^R \int_0^1 (1-\theta)^t (U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0) \cdot D^2\eta^*(U_0^\Delta(r, n\Delta t + 0)$$
$$+ \theta(U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0))) \cdot (U^\Delta(r, n\Delta t - 0)$$
$$- U_0^\Delta(r, n\Delta t + 0))d\theta\, dr$$
$$\leqq C.$$

Since ${}^tV \cdot D^2\eta^* \cdot V \geqq \frac{1}{C}|V|^2$, we get

$$\sum_n \int_1^R |U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)|^2 dr \leqq C,$$

from which it follows that

$$\sum_n \int_1^R |U_0^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)|^2 dr \leqq C. \tag{9}$$

Now let us go back to (8). By Lemma 4.1 of [7], we see

$$|L_1\phi| \leqq C\|\phi\|_{C(\Omega)},$$

$$|L_3\phi| \leqq C\|\phi\|_{C(\Omega)} \int_0^T \sum_{\text{shock}} (\sigma[\eta^*]_0 - [q^*]_0)dt \leqq C'\|\phi\|_{C(\Omega)},$$

$$\begin{aligned}|L_{21}\phi| \leqq C\|\phi\|_{C(\Omega)} \sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} \int_0^1 (1-\theta)^t (U^\Delta(r, n\Delta t - 0) \\ - U_0^\Delta(r, n\Delta t + 0)) \cdot D^2\eta^*(U_0^\Delta(r, n\Delta t + 0) + \theta(U^\Delta(r, n\Delta t - 0) \\ - U_0^\Delta(r, n\Delta t + 0))) \cdot (U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0))d\theta\, dr \\ \leqq C'\|\phi\|_{C(\Omega)},\end{aligned}$$

$$|L_{22}\phi| \leqq C\Delta t \sum_n \int_1^R |\widetilde{H}|\, |\phi| dr \leqq C'\|\phi\|_{C(\Omega)},$$

$$\begin{aligned}|L_{23}\phi| &\leqq \sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} |\phi(r, n\Delta t) - \phi_{n,k}|\, |\eta(U^\Delta(r, n\Delta t - 0)) \\ &\qquad - \eta(U_0^\Delta(r, n\Delta t + 0))| dr \\ &\leqq C\Delta^\alpha \|\phi\|_{C^\alpha(\Omega)} \sum_n \left( \int_1^R |U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)|^2 dr \right)^{1/2} \\ &\leqq C'\Delta^{\alpha - 1/2} \|\phi\|_{C^\alpha(\Omega)} \left( \sum_n \int_1^R |U^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)|^2 dr \right)^{1/2} \\ &\leqq C''\Delta^{\alpha-1/2}\|\phi\|_{C^\alpha(\Omega)}\end{aligned}$$

for $\frac{1}{2} < \alpha < 1$.

$$|L_4\phi| \leqq C\|\phi\|_{C(\Omega)}.$$

On the other hand, since $0 \leqq \rho \leqq C$ and $|u| \leqq C$, $L_1 + L_2 + L_3 + L_4$ is bounded in $W^{-1,\beta}(\Omega)$ $(\beta > 1)$. Hence $L_1 + L_2 + L_3 + L_4$ is relatively compact in $H^{-1}_{\text{loc}}(\Omega)$ by the argument of Ding et al [2], [3]. Finally we see

$$|L_0\phi| \leqq C\Delta\|\phi\|_{H^1(\Omega)}.$$

Therefore $L_0 + L_1 + L_2 + L_3 + L_4$ is relatively compact in $H^{-1}_{\text{loc}}(\Omega)$. (See [3], p.78.) This completes the proof.

PROPOSITION 2. *There exists a constant $C$ independent of small $\Delta$ such that*

$$\sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} |U_0^\Delta(r, n\Delta t - 0) - U_{0,n,k}^\Delta|^2 dr \leqq C,$$

*where the summation is taken over $n, k$ such that $n + k$ is odd and*

$$U^{\Delta}_{0,n,k} = \frac{1}{2\Delta} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} U^{\Delta}_0(r, n\Delta t - 0)dr.$$

*(When $n$ is odd and $k = 2$, then $1 + (k - 1)\Delta$ stands for 1 and $U_{0,n,2} = \frac{1}{3\Delta} \int_1^{1+3\Delta} U^{\Delta}_0(r, n\Delta t - 0)dr$.)*

*Proof.* Let us recall (9) obtained in the proof of Proposition 1. There

$$U^{\Delta}_0(r, n\Delta t + 0) = U^{\Delta}_{n,k} = \frac{1}{2\Delta} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} U^{\Delta}(s, n\Delta t - 0)ds$$

for $1 + (k - 1)\Delta \leqq r < 1 + (k + 1)\Delta$. On the other hand, since $U^{\Delta} - U^{\Delta}_0 = O(\Delta t)$, we have $U^{\Delta}_{n,k} - U^{\Delta}_{0,n,k} = O(\Delta t)$. Therefore

$$\begin{aligned}
&\sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} |U^{\Delta}_0(r, n\Delta t - 0) - U^{\Delta}_{0,n,k}|^2 dr \\
&\leqq 2 \sum_n \int_1^R |U^{\Delta}_0(r, n\Delta t - 0) - U^{\Delta}_0(r, n\Delta t + 0)|^2 dr \\
&\quad + 2 \sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} |U^{\Delta}_{n,k} - U^{\Delta}_{0,n,k}|^2 dr \leqq C + C'\Delta \leqq C''.
\end{aligned}$$

This completes the proof.

Proposition 3. *We have*

$$\sum_n \int_{(n-1)\Delta t}^{n\Delta t} \int_1^R |U^{\Delta}_0(r, t) - U^{\Delta}_0(r, n\Delta t - 0)|^2 dr\, dt = O(\Delta).$$

*Proof.* From Proposition 2, it suffices to show, for $(n - 1)\Delta t \leqq t < n\Delta t$, that

$$\begin{aligned}
&\int_{1+(k-1)\Delta}^{1+(k+1)\Delta} |U^{\Delta}_0(r, t) - U^{\Delta}_0(r, n\Delta t - 0)|^2 dr \\
&\quad \leqq C \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} |U^{\Delta}_0(r, n\Delta t - 0) - U^{\Delta}_{0,n,k}|^2 dr,
\end{aligned}$$

where $n + k$ is odd. This is reduced to proving for the solution $U(x, t) = U_0(x, t; U_L, U_R)$ of the Riemann problem $(5)_0{}'$ (7) that

$$\int_{-\Delta x}^{\Delta x} |U(x, \Delta t) - U(x, t)|^2 dx \leqq C \int_{-\Delta x}^{\Delta x} |U(x, \Delta t) - \overline{U}|^2 dx,$$

where $\overline{U} = \frac{1}{2\Delta x}\int_{-\Delta x}^{\Delta x} U(x,\Delta t)dx$, for $0 \leqq t < \Delta t$. (Recall that we are assuming that $\Delta x \geqq 2\Lambda_0 \Delta t$.) This can be shown as follows. First we note

$$\begin{aligned}&\int_{-\Delta x}^{\Delta x} |U(x,\Delta t) - U(x,t)|^2 dx \\ &\leqq 2\int_{-\Delta x}^{\Delta x} |U(x,\Delta t) - \overline{U}|^2 dx + 2\int_{-\Delta x}^{\Delta x} |U(x,t) - \overline{U}|^2 dx.\end{aligned}$$

Since $U$ is of the form $U(x,t) = V(x/t)$, we have

$$\begin{aligned}&\int_{-\Delta x}^{\Delta x} |U(x,t) - \overline{U}|^2 dx = \int_{-\Delta x}^{\Delta x} |V(x/t) - \overline{U}|^2 dx \\ &= \frac{t}{\Delta t}\int_{-\frac{\Delta t}{t}\Delta x}^{\frac{\Delta t}{t}\Delta x} |V(y/\Delta t) - \overline{U}|^2 dy = \frac{t}{\Delta t}\left(\int_{-\Delta x}^{\Delta x} + \int_{\Delta x}^{\frac{\Delta t}{t}\Delta x} + \int_{-\frac{\Delta t}{t}\Delta x}^{-\Delta x}\right) \\ &= \frac{t}{\Delta t}\left(\int_{-\Delta x}^{\Delta x} |V(y/\Delta t) - \overline{U}|^2 dy + \left(\frac{\Delta t}{t}\Delta x - \Delta x\right)\left(|U_R - \overline{U}|^2 + |U_L - \overline{U}|^2\right)\right).\end{aligned}$$

On the other hand, since $\Delta x/\Delta t \geqq 2\Lambda_0$, we have

$$\begin{aligned}&\int_{-\Delta x}^{\Delta x} |U(x,\Delta t) - \overline{U}|^2 dx = \int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} + \int_{\frac{1}{2}\Delta x}^{\Delta x} + \int_{-\Delta x}^{-\frac{1}{2}\Delta x} \\ &= \int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} |U(x,\Delta t) - \overline{U}|^2 dx + \frac{\Delta x}{2}\left(|U_R - \overline{U}|^2 + |U_L - \overline{U}|^2\right) \\ &\geqq \frac{\Delta x}{2}\left(|U_R - \overline{U}|^2 + |U_L - \overline{U}|^2\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{-\Delta x}^{\Delta x} |U(x,t) - \overline{U}|^2 dx \\ &\leqq \frac{t}{\Delta t}\int_{-\Delta x}^{\Delta x} |V(y/\Delta t) - \overline{U}|^2 dy + 2\left(1 - \frac{t}{\Delta t}\right)\int_{-\Delta x}^{\Delta x} |U(x,\Delta t) - \overline{U}|^2 dx \\ &\leqq 2\int_{-\Delta x}^{\Delta x} |U(x,\Delta t) - \overline{U}|^2 dx.\end{aligned}$$

Consequently

$$\int_{-\Delta x}^{\Delta x} |U(x,\Delta t) - U(x,t)|^2 dx \leqq 6\int_{-\Delta x}^{\Delta x} |U(x,\Delta t) - \overline{U}|^2 dx.$$

This completes the proof.

The estimate given in Proposition 3 will play an important role in the proof of Theorem. However we could not understand the proof of the corresponding statement in [3] (pp.80–81, $J_2 \to 0$). Thus we have proposed an alternative proof.

By Proposition 1 we can obtain the compactness frame work result of the approximate solutions, that is, there is a sequence in $U^\Delta$ such that $\Delta_j \to 0$, $U^{\Delta_j} \to U = (\rho, m)$ almost everywhere as $j \to \infty$. (See [1], [2].) Since $U^\Delta \in \Sigma(B^{00})$, we have $0 \leqq \rho \leqq C$, $|m| \leqq C\rho$. We must show that the limit $U$ is a weak solution.

Let $\Phi(r,t) = {}^t(\phi_1, \phi_2)$ be a test function in $C_0^\infty([1,R) \times [0,T))$ such that $\phi_{2|r=1} = 0$. Putting

$$J = \int_0^T \int_1^\infty (\Phi_t \cdot U^\Delta + \Phi_r \cdot f(U^\Delta) + \Phi \cdot H(r, U^\Delta))dr\,dt + \int_1^\infty \Phi(r,0)U^\Delta(r,0)dr,$$

we will show that $J \to 0$ as $\Delta \to 0$. We put $J = J_1 + J_2$, where

$$J_1 = \int_0^T \int_1^R (\Phi_t U_0^\Delta + \Phi_r f(U_0^\Delta) + \Phi H(r, U_0^\Delta))dr\,dt + \int_1^R \Phi(r,0)U_0^\Delta(r,0)dr,$$

$$J_2 = \sum_n \int_{n\Delta t}^{(n+1)\Delta t} \int_1^R \Big(\Big(\Phi_t + \Phi_r \int_0^1 Df(U_0^\Delta + \theta\widetilde{H}(r, U_0^\Delta, t - n\Delta t)(t - n_\Delta t))d\theta\Big)\widetilde{H}(r, U_0^\Delta, t - n\Delta t)(t - n\Delta t) + \Phi(H(r, U^\Delta) - H(r, U_0^\Delta))\Big)dr\,dt.$$

We see

$$\begin{aligned}|J_2| &\leqq \Delta t \int_0^T \int_1^R |\Phi_t + \Phi_r \int_0^1 Df(U_0^\Delta + \theta\widetilde{H})d\theta|\,|\widetilde{H}|dr\,dt + \int_0^T \int_1^R |\Phi|\,|H(r,U^\Delta) - H(r,U_0^\Delta)|dr\,dt \\ &\leqq C\Delta t + C\int_0^T \int_1^R |H(r,U^\Delta) - H(r,U_0^\Delta)|dr\,dt = O(\Delta).\end{aligned}$$

On the other hand,

$$\begin{aligned}J_1 &= \sum_n \int_1^R \Phi(r, n\Delta t)(U_0^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0))dr + \int_0^T \int_1^R \Phi H(r, U_0^\Delta)dr\,dt \\ &= J_{11} + J_{12},\end{aligned}$$

where

$$J_{11} = \sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} (\Phi - \Phi_{n,k})(U_0^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0))dr,$$

$$J_{12} = \sum_{n,k} \Phi_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} (U_0^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0))dr$$

$$+ \int_0^T \int_1^R \Phi H(r, U_0^\Delta) dr\, dt,$$

$$\Phi_{n,k} = \Phi(1 + k\Delta, n\Delta t).$$

The summation is taken over $n, k$ such that $n + k$ is odd. From (9) it follows that

$$|J_{11}| \leqq C\Big(\sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} |\Phi - \Phi_{n,k}|^2 dr\Big)^{1/2} \times$$

$$\Big(\sum_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} |U_0^\Delta(r, n\Delta t - 0) - U_0^\Delta(r, n\Delta t + 0)|^2 dr\Big)^{1/2} = O(\Delta^{1/2}).$$

On the other hand,

$$J_{12} = -\Delta t \sum_{n,k} \Phi_{n,k} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} \widetilde{H}(r, U_0^\Delta(r, n\Delta t - 0), \Delta t) dr$$

$$+ \int_0^T \int_1^R \Phi H(r, U_0^\Delta) dr\, dt$$

$$= J_{121} + J_{122} + J_{123},$$

where

$$J_{121} = \sum_{n,k} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} (\Phi(r,t) - \Phi_{n,k}) H(r, U_0^\Delta(r,t)) dr\, dt,$$

$$J_{122} = \sum_{n,k} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} \Phi_{n,k}(H(r, U_0^\Delta(r,t)) - \widetilde{H}(r, U_0^\Delta(r,t), \Delta t)) dr\, dt,$$

$$J_{123} = \sum_{n,k} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} \Phi_{n,k}(\widetilde{H}(r, U_0^\Delta(r,t), \Delta t)$$

$$- \widetilde{H}(r, U_0^\Delta(r, n\Delta t - 0), \Delta t)) dr\, dt.$$

We see $J_{121} = O(\Delta)$ clearly, $J_{122} = O(\Delta)$ since $H(r,U) - \widetilde{H}(r, U, \Delta t) = O(\Delta t)$, and

$$|J_{123}| \leqq C \sum_{n,k} \int_{(n-1)\Delta t}^{n\Delta t} \int_{1+(k-1)\Delta}^{1+(k+1)\Delta} |U_0^\Delta(r,t) - U_0^\Delta(r, n\Delta t - 0)| dr\, dt$$
$$= O(\Delta^{1/2})$$

from Proposition 3. Summing up, we see $J \to 0$. Thus we have obtained the following conclusion.

THEOREM. *There exists a weak solution* $U$ *of* (5)′ (4) (3) *on* $1 \leqq r < +\infty$, $0 \leqq t < T$ *such that* $0 \leqq \rho \leqq C$, $|m| \leqq C\rho$, *provided that* $\rho^0$ *and* $u^0$ *are bounded and* $\rho^0 \geqq 0$ *has compact support. Here* $T$ *is a positive number depending upon* $\rho^0$, $u^0$, $A$, $\gamma$, *and* $M$.

Unfortunately the solution constructed in this paper is temporally local. To establish the global existence remains an open problem. Although there are blowing up solutions for the Euler-Poisson equation of self-gravitating gaseous stars (see [5]), we conjecture that in the case of this exterior problem for the atmosphere any weak solution might be temporally global. In fact, for the case where $\gamma = 1$ (the isothermal case), the existence of global solutions was established by K. Mizohata et al [6] by using the Glimm's method. But it seems that our scheme should be replaced by another one if we wish to get global solutions. So, our investigation is temporary and further studies are expected.

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