TRANSFORMATIONS OF HEEGAARD DIAGRAMS
CORRESPONDING TO THOSE OF
HEEGAARD CUT DIAGRAMS

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1. Introduction

Heegaard cut diagrams are helpful for studies of the Heegaard diagrams of 3-manifolds. Sometimes Heegaard cut diagram is called as Heegaard diagram but we distinguish clearly between them. Transformations of Heegaard cut diagrams are obtained in [5] by applying the DS-deformations (Ikeda, Yamashita, Yokoyama; [4]) for DS-diagrams (Ikeda, Inoue; [1], Ishii; [2]). A transformation of Heegaard cut diagrams is carried out as a 3-regular graph in a plane, while meridian-longitude system in the Heegaard-surface of Heegaard-handlebody is done, corresponding to that of Heegaard diagram. Therefore, we have searched for the transformations of Heegaard diagrams corresponding to those of Heegaard cut diagrams. The answers reach the simple basic transformations in the theorem 1 given in §2. §3 deals with the definitions concerning handlebodies, Heegaard splittings, diagrams, Heegaard cut diagrams and so forth. §4 states the transformations of Heegaard cut diagrams and the methods of those transformations by DS-deformations. §5 gives the proofs of theorem 1. §6 gives the theorem 2 that describes the operations used for the transformations of Heegaard diagrams and DS-deformations used for those of Heegaard cut diagrams. §7 gives some examples to them. In this paper, we use many figures to understand.

Throughout this paper we will be considering the piecewise linear standpoint. ∂X, Int(X), Cl(X) indicates the boundary, interior, closure of a point set X, respectively. From now on, notation $M^3$ denotes a closed, connected orientable 3-manifold unless otherwise stated.

2. Transformations of Heegaard diagrams

We begin with the following basic theorem.

Theorem 1. Let $U \cup V$ be a Heegaard splitting of $M^3$. $U$, $V$ is a Heegaard-handlebody, respectively. Let $(U; m, l) \cup (V; l, m)$ be Heegaard diagrams associated with $U \cup V$. $(m, l)$ is a meridian-longitude system of $(U; m, l)$.

(1) Let the following figure $U1-A$ be a part of $(U; m, l)$. The longitudes $\{l_0, \ldots, l_{\alpha}\}$ go around side by side on the two handles $h_m$ and $h_n$. The longitudes $\{l_0, \ldots, l_{\gamma}\}, \{l_0, \ldots, l_{\beta}\}$ go around on $h_m$, $h_n$, respectively. Then there exists a transformation of $(U; m, l)$ so that $\{l_0, \ldots, l_{\alpha}\}$ go around on $h_m$ (not on $h_n$). $\{l_0, \ldots, l_{\gamma}\}$ go around on both the $h_m$ and $h_n$, and $\{l_0, \ldots, l_{\beta}\}$ do not change the way of running. Therefore there exists a transformation from $U1-A$ into $U1-B1$ of $(U; m, l)$.

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1 This research is supported by the Mathematical Economics Study Group of Niigata Sangyou University.
(1') Let $V1-A'$ be a part of $(V; l, m)$. $V1-A'$ is the dual part of $U1-A$. If the longitude $m_i, m_j$ crosses the meridians $(l_{n_1}, \ldots, l_{n_p}, l_{i_1}, \ldots, l_{i_q})$, $(l_{i_1}, \ldots, l_{i_q}, l_{i_1}, \ldots, l_{i_q})$, respectively, then there exists a transformation from $V1-A'$ into $V1-B'$ of $(V; l, m)$ so that $m_i$ does not change the way of running, and here $m_j$ comes to cross $(l_{n_1}, \ldots, l_{i_q}, l_{i_1}, \ldots, l_{i_q})$.

$m_i, m_j, l_{i_k}(k = 1, \ldots, l)$ are drawn heavily.

$U1-A$

$V1-A'$
(2) Again let $U_1-A$ be the same figure as in (1). Let $h_{m_A}$ be a new handle and $m_A$, a new meridian shown in $U_1-B_3$. Then there exists a transformation from $U_1-A$ into $U_1-B_3$ of $(U; m, l)$, so that $\{l_1, \ldots, l_p\}$, $\{l_{\hat{h}}, \ldots, l_{\hat{g}}\}$ do not change the way of running, respectively, a new longitude $l_A$ goes around on $h_{m_1}, h_{m_j}$ and $h_{m_A}$, $\{l_{\hat{h}}, \ldots, l_{\hat{g}}\}$ go around on $h_{m_A}$.
(2') Again let $V1-A'$ be the same figure as in (1'). Let $h_A'$ be a new handle and $l_A$ a new meridian shown in $V1-B'$. Then there exists a transformation from $V1-A'$ into $V1-B'$ of $(V; l, m)$ so that $m_i, m_j, m_A$ cross the meridians $(l_i, \cdots, l_p, l_A), \{l_i, \cdots, l_i, l_A\}, \{l_i, \cdots, l_i, l_A\}$, respectively.

(3) Let $U2-A$ be a part of $(U; m, l)$. The longitudes $(l_{i_1}, \cdots, l_{i_q})$ go around side by side on the handle $h_{m_{i_1}}$ twice. The longitudes $(l_{i_1}, l_{i_1}, \cdots, l_{i_{p_q}}), \{l_{i_1}, \cdots, l_{i_{p_q}}\}$ go around on $h_{m_{i_1}}$ only once, respectively. Let $h_{m_{i_1}}, h_{m_{i_2}}$ be a handle shown in $U2-B$, respectively. Let $m_{i_1}, m_{i_2}$ be a meridian in $h_{m_{i_1}}, h_{m_{i_2}}$, respectively. Then there exists a transformation so that $(l_{i_1}, \cdots, l_{i_q})$ go around on $h_{m_{i_1}}$, the longitude $l_A$ goes around on $h_{m_{i_1}}$ twice and $h_{m_{i_2}}$ once. Other longitudes in $U2-A$ come to go around on $h_{m_{i_1}}$. Therefore, there exists a transformation from $U2-A$ into $U2-B$ of $(U; m, l)$.

(3') Let $V2-A'$ be the dual part of $U2-A$. The longitude $m_i$ crosses $(l_{i_1}, \cdots, l_{i_q})$ side by side twice, and $(l_{i_1}, l_{i_1}, \cdots, l_{i_{p_q}}), \{l_{i_1}, \cdots, l_{i_{p_q}}\}$ once. Let $h_A'$ be a new handle and $l_A$ a new meridian shown in $V2-B'$. Then there exists a transformation from $V2-A'$ into $V2-B'$ of $(V; l, m)$ so that $m_{i_1}, m_{i_2}$ cross $(l_{i_1}, \cdots, l_{i_{p_q}}, l_A, l_{i_1}, l_{i_1}, \cdots, l_{i_{p_q}})$, $(l_{i_1}, \cdots, l_{i_q}, l_A)$, respectively.
(4) Let $U3-A$ be a part of $(U; m, l)$. If the meridian $m$ crosses the longitude $l$, then there exists a transformation from $U3-A$ into $U3-B$ of $(U; m, l)$, so that the handle $h_m$ and longitude $l$ are erased.

(4') Let $V3-A'$ be the dual part of $U3-A$. If the longitude $m$ goes around on the handle $h_i'$ once, then there exists a transformation from $V3-A'$ into $V3-B'$ of $(V; l, m)$, so that $h_i'$ and $m$ are erased.
Let $U^4 - A$ be a part of $(U; m, l)$. If the longitude $l$ goes around on the two handles $h_{m_i}$ and $h_{m_j}$ once, then there exists a transformation from $U^4 - A$ into $U^4 - B$ of $(U; m, l)$, so that $l$ and $h_{m_j}$ are erased, and the longitudes that went around on $h_{m_i}$ come to go around on $h_{m_i}$.

Let $V^4 - A'$ be the dual part of $U^4 - A$. If the two longitudes $m_i$ and $m_j$ go around on the handle $h_i'$, then there exists a transformation from $V^4 - A'$ into $V^4 - B'$ of $(V; l, m)$, so that $m_j$ and $h_i'$ are erased.
(6) \( (6') \) respectively. Let \( U_5 - A \) \((V_5 - A') \) resp. be a part of \((U ; m, l) \((V ; l, m) \) resp.\). If the longitude \( l_i \) \((m_j \) resp.\) crosses the meridian \( m_j \) \((l_i \) resp.\), turns back to \( m_j \) \((l_i \) resp.\) and crosses \( m_j \) \((l_i \) resp.\) again, then there exists a transformation from \( U_5 - A \) \((V_5 - A') \) resp. into \( U_5 - B \) \((V_5 - B') \) resp. of \((U ; m, l) \((V ; l, m) \) resp.\), so that \( l_i \) \((m_j \) resp.\) does not cross \( m_j \) \((l_i \) resp.\).

Remark 1. \( V_3 - B' \) is a part of ball and those longitudes \( m_1, \cdots, m_{n-1} \) run on its boundary. \((U_4 - A) \cup (V_4 - A') \) is a special case of \((U_1 - A) \cup (V_1 - A') \) when \( \{ l_{i_1}, \cdots, l_{i_l} \} = \{ l \} \).

(1), \( (4') \), (6) and \( (6') \) are already well-known transformations. That is, we have obtained such transformations from those of Heegaard cut diagrams given in §4.

3. Handlebodies, Heegaard splittings, diagrams and Heegaard cut diagrams

We begin with the definition of a handlebody.

Definition 1. Let \{\( D_i, \cdots, D_n \)\} be mutually disjointed 2-disks and \( h_i = D_i \times [0, 1] \) \((i = 1, \cdots, n)\). A handlebody \( H \) of genus \( n \) is a 3-ball (cube) \( B^3 \) with \( n \) handles \{\( h_i \)\} so that the result of attaching \( h_i \) with homeomorphisms throws \( 2n \) disks \( D_i \times 0, D_i \times 1 \) onto \( 2n \) disjointed 2-disks on
$\partial B^3$. $H$ is represented as $B^3 + \cup_{i=1}^n \{ h_i \}$ where $B^3 \cap h_i = \partial B^3 \cap \partial h_i = \{ D_i \times 0, D_i \times 1 \}$.

We note that $\partial H$ is an orientable or nonorientable closed surface of Euler characteristic $2 - 2n$ according as $H$ is orientable or nonorientable.

**Definition 2.** Let $H$ be a genus $n$ handlebody and $\{ D_i \}$ $(i = 1, \ldots, n)$, mutually disjointed properly embedded 2-disks in $H$. If the $\text{Cl}(H - \{ D_1 \cup \cdots \cup D_n \})$ becomes 3-ball, then the collection $\{ D_i \}$ $(i = 1, \ldots, n)$ is called a complete system of meridian disks of $H$ and $\{ \partial D_i \}$ a complete system of meridian circles of $\partial H$.

Note that $\{ D_1 \cup \cdots \cup D_n \}$ cuts $\partial H$ into 2-sphere with $2n$ holes.

**Definition 3.** In the following, let $H$ be an orientable genus $n$ handlebody with the same presentation as in Def. 1.

1) **Handle sliding**; let $h_i$, $h_j$ be a handle of $H$ shown in Fig. 3-1-1, respectively. By an ambient isotopy of $H$, keeping $D_i \times 0$ fixed, and sliding $D_i \times 1$ along the directions of the lines in $\partial (B^3 + h_i)$ as shown in Fig. 3-1-1, 3-1-2, $h_i$ goes over the $h_j$ and turns back to the first place. This operation is called a handle sliding of $h_i$ about $h_j$. 

![Fig. 3-1-1](image1)

![Fig. 3-1-2](image2)
(2) **Handles combining**: let $\hat{D}_i, \hat{D}_j$ be a disk in the foot of $\partial h_i, \partial h_j$ shown in Fig. 3–2–1, respectively. Gluing together $h_i$ and $h_j$ by an orientation-reversing homeomorphism $f: \hat{D}_i \to \hat{D}_j$, a handlebody with the deformed part shown in Fig. 3–2–2 is obtained. This operation is called as **handles combining with** $h_i$ **and** $h_j$.

![Fig. 3–2–1](image)

(3) **Longitudes combining**: suppose that two circles (longitudes) $m_i$ and $m_j$ run on $\partial V$ as shown in V1–A' in the theorem 1. Let $m_i$ ($m_j$ resp.) be a part of $m_i$ ($m_j$ resp.) that runs from $l_{i_1}$ to $l_{i_j}$. Let $N(m_i)$ ($N(m_j)$ resp.) be a regular neighborhood of $m_i$ ($m_j$ resp.) in $m_i$ ($m_j$ resp.). Again we put the former label $m_i$ ($m_j$ resp.) on $N(m_i)$ ($N(m_j)$ resp.). Let $m_{ij}$ ($m_{ji}$ resp.) = $Cl(m_i - m_j)$ ($Cl(m_j - m_i)$ resp.). Then $m_{ij}, m_{ji}, m_i$ and $m_j$ become four segments. $m_i \cup m_j$ ($m_i \cup m_j$ resp.) construct $m_i$ ($m_j$ resp.). If we pile $m_i$ on $m_j$, then $\theta$ curve is obtained from $m_i \cup m_j$. It consists of three segments $m_{ij} \cup m_{ji} \cup m_{ii}$ where $m_{ii}$ is the common segment by piling $m_i$ on $m_j$. This operation is called as **longitudes combining with** $m_i$ **and** $m_j$.

(4) **Longitude changing**: let $m_{i2}$ and $m_{j2}$ be the same as those in (3). Then $m_i \cup m_j$ is a circle. Sliding $m_{i2} \cup m_{j2}$ so that $m_{i2} \cup m_{j2}$ and $m_i$ run side by side from $l_{i_1}$ to $l_{i_2}$ and it does not intersect $m_i$. Next by reorienting, it becomes $m_i$ in V1–BI'. This operation is called as **a longitude changing of** $m_i$.

(5) **Handle adding**: let $D_A$ be a new disk and $h_A = D_A \times [0, 1]$. Adding a new handle $h_i$ to
$H$ by the same operation as in Def. 1, a genus $n+1$ handlebody $H' = B^3 + \cup_{i=1}^{\infty} \{ h_i \} + h_A$ is obtained. This operation is called a handle $h_A$ adding to $H$.

(6) **Longitude $m_A$ adding to $m_i$ and $m_j$**; let $\theta$ curve be the same as that in (3). We carry out the handle $h_A'$ added to $V$. We construct $m_i$, $m_j$, $m_A$ in $V_1 - B_3'$ in the theorem 1 from $\theta$. This operation is called a longitude $m_A$ adding to $m_i$ and $m_j$.

(7) **Handle gluing**; let $\tilde{D}_i$, $\tilde{D}_j$ be disks in the feet of $\partial h_i$ shown in Fig. 3–7–1, respectively. Gluing $h_i$ itself by an orientation-reversing homeomorphism $f : \tilde{D}_i \to \tilde{D}_j$, a handlebody with the deformed part shown in Fig. 3–7–2 is obtained. This operation is called as a handle gluing of $h_i$.

By handle gluing of $h_i (i = 1, \cdots, n)$ of $H$ and an ambient isotopy of $H$, a genus $n$ handlebody $H$ is deformed such as shown in Fig. V1–A'. Therefore, it will also be called as a genus $n$ solid torus.

(8) **Longitude gluing**; suppose that circle (longitude) $m_i$ runs on $\partial V$ as shown in V2–A’ in the theorem 1. If we cut off $m_i$ at $\{ D_i, D_j \}$ (\( \partial D_i = l_i, \partial D_j = l_j \)), then four segments arise in $\partial V$. Let $m_{i1}$ $(m_{i2}$ resp.) be the segment that intersects the meridians $\{ l_i, \cdots, l_i \}$. Let $N(m_{i1}$ $m_i$) $(N(m_{i2}$ $m_i$) resp.) be a regular neighborhood of $m_{i1}$ $(m_{i2}$ resp.) in $m_i$. Again we put the former label $m_{i1}$ $(m_{i2}$ resp.) on $N(m_{i1}$, $m_i$) $(N(m_{i2}$, $m_i$) resp.). Let $\{ m_{i1}$, $m_{i2} \} = CL(m_i - m_{i1} \cup m_{i2})$. Then $m_{i1}$, $m_{i2}$, $m_{i1}$ and $m_{i2}$ become four segments. $m_{i1}$ $m_{i2}$ $m_{i1}$ $m_{i2}$ construct $m_i$. If we pile $m_{i1}$ on $m_{i2}$ in $m_i$, then a curve of which shape is $\cdots$, is obtained. This operation is called as a longitude gluing of $m_i$.

(9) **Longitude adding to $m_i$**; let the curve be the same as that in (8). Let $V_2 - B'$ be a handle $h_A'$ added to $V_2 - A'$. We construct $m_{i1}$, $m_{i2}$ in $V_2 - B'$ from this curve. This operation is called a longitude adding to $m_i$.

(10) **Handle cutting (cancellation) by a meridian disk**; this operation is cutting off a handle $h_i$ at a meridian disk $D_i \times r$ ($0 \leq r \leq 1$) in $h_i$. The result of this gives a genus $n-1$ handlebody.
\[ H' = B^3 + \{ h_i \cup \cdots \cup h_{i-1} \cup h_{i+1} \cup \cdots \cup h_n \} \]. This operation is called a handle cutting (cancellation) of \( h_i \) by \( D_i \times r \).

(11-1) Handlebody cutting by an annulus; let \( A \) be an annulus. Suppose that \( A = c \times [0, 1] \) where \( c \) is a circle. We embed \( A \) in \( H \) so that \( \text{Int} (A) \subset \text{Int} (H) \) and \( \partial A \subset \partial H \). By cutting off \( H \) at \( A \), a genus \( n+1 \) handlebody is obtained. This operation is called a handlebody cutting of \( H \) by an annulus \( A \).

The result of the above operation is the same as that of the handle \( h_i \) added to \( H \), i.e., genus \( n+1 \) handlebody \( H' = B^3 + \bigcup_{i=1}^{n} \{ h_i \} + h_i \) is obtained.

(11-2) Handlebody cutting by a disk; let \( D \) be a properly embedded 2-disk in a handlebody \( H \) so that \( \text{Int} (D) \subset \text{Int} (H) \). Note that \( \partial D \) or a part of \( \partial D \) containing in \( \text{Int} (H) \) is permitted. This operation is cutting off \( H \) at \( D \). It is called as a handlebody cutting of \( H \) by a disk \( D \).

Note that if the disk \( D \) is chosen as a meridian disk in a handle \( h_i \) of \( H \), this operation becomes the handle cutting of \( h_i \) by \( D \).

(12) Handle crushing and cutting; let \( D_i \times \frac{1}{2} \) be a meridian disk in the handle \( h_i \) of \( H \). By crushing \( h_i \) at \( D_i \times \frac{1}{2} \) so that \( D_i \times \frac{1}{2} \) contracts the center point \( p \) of the meridian disk, a deformed handlebody shown in Fig. 3-12 is obtained. Next by cutting off this at \( p \), genus \( n-1 \) handlebody \( H' = B^3 + \{ h_i \cup \cdots \cup h_{i-1} \cup h_{i+1} \cup \cdots \cup h_n \} \) is obtained. This operation is called handle crushing and cutting of \( h_i \) by a disk.

![Fig. 3-12](image)

The result of the above operations is the same as that of the handle cutting but circles in \( \partial H \) take different operations corresponding to those of (10) and (12).

(13) Meridian sliding; this operation is sliding a meridian \( m_i \) by an ambient isotopy of \( \partial H \). It is called as a meridian \( m_i \), sliding of \( \partial H \).

Next we state the definitions of the Heegaard splitting and diagram as follows.

**Definition 4.** A closed, connected 3-manifold \( M^3 \) is represented with a union of two handlebodies \( H_1, H_2 \) in \( M^3 \); \( M^3 = H_1 \cup H_2 \) so that \( H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = \partial H_1 = \partial H_2 \). \( \partial H_i \) \( (= \partial H_2) \) is a closed surface of genus \( n \). Let the surface be \( F \). \( H_1 \) (\( H_2 \) resp.) \( F \) are orientable or nonorientable according as \( M^3 \) is orientable or nonorientable. A triplet \( (H_1, H_2, F) \) or \( M^3 = H_1 \cup H_2 \) is called a Heegaard splitting (H-splitting) of \( M^3 \) with genus \( n \) and \( H_1 \) (\( H_2 \) resp.), a Heegaard-handlebody (H-handlebody). \( F \) is called a Heegaard-surface (H-surface) and the integer \( n \) \( (\geq 0) \), Heegaard genus (H-genus).
Definition 5. Suppose \((H_1, H_2, F)\) is a genus \(n\) \(H\)-splitting of \(M^3\). Let \(\{D_i, \ldots, D_n\}, \{D'_i, \ldots, D'_n\}\) be a complete system of meridian disks of \(H_1, H_2\), respectively and \(\{m\} = \{m_1, \ldots, m_n\} = \{\partial D_i, \ldots, \partial D_n\}\), \(\{l\} = \{l_1, \ldots, l_n\} = \{\partial D'_i, \ldots, \partial D'_n\}\). Then \((H_1; m, l) ((H_2; l, m)\) resp.) is called a genus \(n\) Heegaard diagram (\(H\)-diagram) associated with \((H_1, H_2, F)\). \(\{m, l\} ((l, m)\) resp.) is called a meridian-longitude system of \((H_1; m, l) ((H_2; l, m)\) resp.), respectively.

Fig. 5 shows the genus \(n\) \(H\)-diagram \((H_1; m, l) ((H_2; l, m)\) resp.) of \((H_1, H_2, F)\) of the 3-sphere \(S^3\). It is called a canonical genus \(n\) \(H\)-diagram.

Let \((H_1; m_1, \ldots, m_n, l_1, \ldots, l_n)\) be a genus \(n\) \(H\)-diagram associated with \((H_1, H_2, F)\). We may assume that \((m_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)\) consists at most of finite points (by an argument of general position).

Definition 6. The number of finite points of \(\{m\} \cap \{l\} = (m_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)\) is called a cross point number with \((H_1; m, l) ((H_2; l, m)\) resp.) The number of finite points of \(l_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)\) \(m_1 \cup \cdots \cup m_n\) resp.) is also called a cross point number with \((H_1; m, l) ((H_2; l, m)\) resp.).

Next we define the Heegaard cut diagram from a Heegaard diagram.

Let \((U, V, F)\) be a genus \(n (\geq 1)\) \(H\)-splitting of \(M^3\) and \(\{D_i, \ldots, D_n\}, \{D'_i, \ldots, D'_n\}\) resp.) a complete system of meridian disks of \(U (V\) resp.). Let \((U; m, l) ((V; l, m)\) resp.) be a genus \(n\) \(H\)-diagram of \((U, V, F)\). \(\{m\} = \{m_1, \ldots, m_n\} = \{\partial D_i, \ldots, \partial D_n\}\) and \(\{l\} = \{l_1, \ldots, l_n\} = \{\partial D'_i, \ldots, \partial D'_n\}\). Suppose each circle \(l_1, m_i\) is oriented. By cutting off \(M^3\) at \(F\), disjointed genus \(n\) handlebodies \(U, V\) put on the former labels are obtained. We put the same label \(F\) on \(\partial U, \partial V\), respectively. Let \(\{j_1, j_2, j_3, \ldots, j_n\}\) be the ordering cross points of \(l_i \cap (m_1 \cup m_2 \cup m_3 \cup \cdots \cup m_n)\) in \(\partial U\). \(\{m_1, \ldots, m_n\}\) \(\partial U\) decomposes each \(l_i\) into edges at those points. We put labels \(j_1(l_1)j_2, j_3(l_2)j_3, \ldots, j_n(l_n)j_1\), in these orders, to these edges in \(l_i\) according to the orientation of \(l_i\) such as \(l_i = j_1(l_1)j_2(l_2)j_3(l_3)\cdots j_n(l_n)j_1\). We may assume that each label \(j_k(l_k)j_{k+1}\) is oriented with the same orientation as \(l_i\). The inverse orientation of \(j_k(l_k)j_{k+1}\) is denoted by \((j_k(l_k)j_{k+1})^{-1}\) or \(j_{k+1}(l_k^{-1})j_k\). Conversely \(\{l_i, \ldots, l_n\}\) \(\partial U\) decomposes each \(m_i\) into edges such as \(m_i = i_1(m_i)i_2 \cdots i_n(m_i)i_1\). By cutting off \(U\) at each \(D_i\), a 3-ball \(B_c^3\) is obtained. \(\partial B_c^3\) is a 2-sphere \(S^2\) and \(n\) pairs of disks \(\{D_i^+, D_i^-\}\) appear in \(S^2\). Since both \(\partial D_i^+\) and \(\partial D_i^-\) are decomposed by the same edges in \(m_i\), they have oriented labels \(i_1(m_i)i_2, i_2(m_i)i_3, \ldots, i_n(m_i)i_1\) in common. Then we
have a planar 3-regular graph described as a diagram over a plane \((= \mathbb{S}^2 - \infty)\) if a point \(\infty \in \mathbb{S}^2\) is designated.

Therefore, we go to the next definition.

**Definition 7.** A planar 3-regular graph

\[
\{(m_i = \partial D_i^+) = i_1(m_i)i_2(m_i)i_3(m_i)\cdots i_n(m_i)i_1, \\
(m_i = \partial D_i^- = i_1(m_i)i_2(m_i)i_3(m_i)\cdots i_n(m_i)i_1, \\
\{j_1(l_j)j_2, j_2(l_j)j_3, \cdots, j_n(l_j)j_1\}\} \ (i, j = 1, \cdots, n)
\]

is called a **Heegaard cut diagram** (H-cut-diagram) associated with \((U; m, l)\) and is described as \(G(m, l)\). In like manner \(G(l, m)\) of \((V; l, m)\) is defined and its expression is

\[
G(l, m) = \{(l_j = \partial D_j^+ = j_1(l_j)j_2(l_j)j_3(l_j)\cdots j_n(l_j)j_1, \\
(l_j = \partial D_j^- = j_1(l_j)j_2(l_j)j_3(l_j)\cdots j_n(l_j)j_1, \\
\{i_1(m_j)i_2, i_2(m_j)i_3, \cdots, i_n(m_j)i_1\}\} \ (i, j = 1, \cdots, n).
\]

Note that the set of vertices \(U_j \cap \{i_1, i_2, i_3, \cdots, i_n\}\) equals to \(U_j \cap \{j_1, j_2, j_3, \cdots, j_n\}\).

Fig. 7 shows the genus \(n\) H-cut-diagram \(G(m, l)\) \((G(l, m)\) resp.) of the canonical H-diagram \((U; m, l)\) \((V; l, m)\) resp.) of the 3-sphere \(S^3\).

![Diagram](image)

**Fig. 7**

**Remark 2.** If the handlebody \(U\) of \((U; m, l)\) is represented such as \(B^3 + \cup_{i=1}^{n} \{h_i\}\) in Def. 1, then there exists the H-cut-diagram \(G(m, l)\) of \((U; m, l)\) in \(\partial B^3\).

Let \(|G(m, l)|\) be the presentation of the underlying space of labels of \(G(m, l)\). Let \(\sigma_1, \cdots, \sigma_p\) be faces that are the connected components of \(\mathcal{C}(\mathbb{S}^2 - |G(m, l)| - \cup_{i=1}^{n} (D_i^+ \cup D_i^-))\). Let the name \(\sigma_i\) of face be its label. Since \(\partial U\) and \(\partial V\) are put on the same label \(F\), there exists the face that should be put on the same label \(\sigma_i\) in \(|G(l, m)|\). \(\{m\} \cap \{l\}\) consists of cross points of \((U; m, l)\) or \((V; l, m)\). Since there exist the same labeled two edges \(i_1(m_i)i_{i+1}\) in \(G(m, l)\), there exist the same labeled two points \(i_k\) in \(G(m, l)\) (see §4 Fig. (1-A) \(\cup (1-A')\)). Since there is \(i_k(i_{k+1})i_{k+1}\) in \(G(l, m)\), there exist \(D_i^+, D_i^-\) such as \(i_k \subset \partial D_i^+\) and \(i_k \subset \partial D_i^-\) in \(|G(l, m)|\). We put the same label \(D_i\) \((D_i^+\text{ resp.})\) on the two disks \(D_i^+, D_i^-\) \((D_i^+, D_i^-\text{ resp.) in }|G(m, l)| \cup |G(l, m)|\\) resp.). Therefore by counting the same labels of the vertices, edges and faces in \(|G(m, l)|\) \(\cup |G(l, m)|\), we have ;
Proposition 1. There exist the same labeled four vertices, the same oriented labeled three edges and the same oriented labeled two faces, i.e., 2-disks or punctured 2-disks\(^2\) in \(|G(m, l)| \cup |G(l, m)|\).

According to the Prop.1, we see that \(G(m, l) \cup G(l, m)\) has the same structure of a DS-diagram without any connectedness (components of a DS-diagram are one and those of \(G(m, l) \cup G(l, m)\), more than one). Furthermore, a definition according to that of DS-diagram of H-cut-diagram is given in [8]. Although we give a definition of the H-cut-diagram, further we can obtain a detailed presentation from the picture of \(G(m, l)\) or \(G(l, m)\) by reading the labels of edges in all \(\partial \sigma\), \(\partial D\), or \(\partial D'\). Conversely, we can draw the picture of a H-cut-diagram from this presentation. See [7].

4. Transformations of Heegaard cut diagrams

Applying the DS-deformations to \(G(m, l) \cup G(l, m)\), we obtain the following transformations of H-cut-diagrams in [5].

Definition 8. We state the transformations of H-cut-diagrams by the following tables 1, 2, 3, 4 and 5, and corresponding figures to them.

Table 1. \(H_0, H\_\text{-}c\text{-}transformation\)

| \(1\)–\(A\) \(\Rightarrow\) \(1\)–\(B1\) | \(H_0(l^-, p^+)\) \(l \geq 1, p \geq 0\) |
| \(1\)–\(A\) \(\Rightarrow\) \(1\)–\(B2\) |
| \(1\)–\(A'\) \(\Rightarrow\) \(1\)–\(B1'\) | \(H_0(l^-, p^{-1})\) |
| \(1\)–\(A'\) \(\Rightarrow\) \(1\)–\(B2'\) |

\(1\)–\(A\), \(1\)–\(A'\), \(1\)–\(B1\), \(1\)–\(B1'\), \(1\)–\(B2\), \(1\)–\(B2'\) denotes the following figure, i.e., the part of a H-cut-diagram, without dotted lines, respectively. Later dotted lines are made use of transformations of H-cut-diagrams. \(1\)–\(A'\), \(1\)–\(B1'\), \(1\)–\(B2'\) is the dual H-cut-diagram to \(1\)–\(A\), \(1\)–\(B1\), \(1\)–\(B2\), respectively.

\(^2\) disk with \(n (\geq 1)\) holes.
The first line in the table 1 is this:

| 1–A ⇒ 1–B1 | Hₜ(l⁻, p⁺) | l ≥ 1, p ≥ 0 |

Notation 1–A ⇒ 1–B1 denotes the transformation from 1–A into 1–B1. This transformation is called $Hₜ(l⁻, p⁺)$-transformation of H-cut-diagram if $l ≥ 1$, $p ≥ 0$.

Transformation from 1–A (1–A' resp.) to 1–B1 (1–B1' resp.) does not change the H-genus but changes the cross point number as many as $|l−p|$. Transformation from 1–A (1–A' resp.) to 1–B2 (1–B2' resp.) does not also change the H-genus but changes the cross point number as many as $|l−q|$.

**Remark 3.** If $p = 0$, then $G(m, l)$ has a wave in a part of 1–A (to know the definition, for instance, see [16]).

**Table 2. $Hₜ$, $Hₜ$-transformation**

<table>
<thead>
<tr>
<th>1–A ⇒ 1–B3</th>
<th>Hₜ(l⁻, 3⁺)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2–A ⇒ 2–B</td>
<td>Hₜ(3⁻, l⁺)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1–B3 ⇒ 1–A</th>
<th>Hₜ(3⁻, l⁺)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2–B ⇒ 2–A</td>
<td>Hₜ(l⁻, 3⁺)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1–A' ⇒ 1–B3'</th>
<th>Hₜ(l⁻, 3⁺)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2–A' ⇒ 2–B'</td>
<td>Hₜ(3⁻, l⁺)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>1–B3' ⇒ 1–A'</th>
<th>Hₜ(3⁻, l⁺)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2–B' ⇒ 2–A'</td>
<td>Hₜ(3⁻, l⁺)</td>
</tr>
</tbody>
</table>
Transformation from 1–A (1–A’ resp.) to 1–B3 (1–B3’ resp.) increases the H-genus, only one and decreases the cross point number as many as \( l - 3 \) (\( l \geq 3 \)), and increases the cross point number, only one (\( l = 2 \)).

Transformation from 2–A (2–A’ resp.) to 2–B (2–B’ resp.) increases the H-genus, only one and decreases the cross point number as many as \( l - 3 \) (\( l \geq 3 \)), and increases the cross point number, only one (\( l = 2 \)).

<table>
<thead>
<tr>
<th>Table 3. ( H^<em>_t, H^</em>_b )-transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>3–A ⇒ 3–B</td>
</tr>
<tr>
<td>3–B ⇒ 3–A</td>
</tr>
<tr>
<td>3–A’ ⇒ 3–B’</td>
</tr>
<tr>
<td>3–B’ ⇒ 3–A’</td>
</tr>
</tbody>
</table>

\( l \geq 2 \)

3–A

3–A'
Transformation from 3–A (3–A’ resp.) to 3–B (3–B’ resp.) decreases the H-genus, only one and the cross point number, as many as $l$.

<table>
<thead>
<tr>
<th>$4$–A $\Rightarrow$ $4$–B</th>
<th>$H_4(2^-)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4$–B $\Rightarrow$ $4$–A</td>
<td>$H_4(2^+)$</td>
</tr>
<tr>
<td>$4$–A’ $\Rightarrow$ $4$–B’</td>
<td>$H_4(2^-)$</td>
</tr>
<tr>
<td>$4$–B’ $\Rightarrow$ $4$–A’</td>
<td>$H_4(2^+)$</td>
</tr>
</tbody>
</table>
4-A

4-\text{A}'

4-B

4-B'
Transformation from 4–A (4–A’ resp.) to 4–B (4–B’ resp.) decreases the H-genus, only one and the cross point number, as many as 2.

<table>
<thead>
<tr>
<th>5–A</th>
<th>5–B</th>
<th>$H_b^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5–A’</td>
<td>5–B’</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>5–B</th>
<th>5–A</th>
<th>$H_b^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5–B’</td>
<td>5–A’</td>
<td></td>
</tr>
</tbody>
</table>

Those transformations do not change the H-genus but decrease the cross point number, as many as 2.

Transformations of tables 1, 2, 3, 4 and 5 will be generally called the $H$-cut-transformations for H-cut-diagrams.

Next we illustrate the methods of H-cut-transformations using the following $DS$-deformations.

**Definition 9.**

1. $G–1 \Rightarrow G–2$ is called $D_5^+$-deformation and $G–2 \Rightarrow G–1$ is called $D_5^-$-deformation. They are generally called $D_5$-deformations.

2. $G–3 \Rightarrow G–4$ is called $D_4^+$-deformation and $G–4 \Rightarrow G–3$ is called $D_4^-$-deformation. They are generally called $D_4$-deformations.
(1) and (2) are generally called the \textit{elementary DS-deformations} for DS-diagram.

(3) $G - 5 \Rightarrow G - 6$ is called $D_{n+2}^+$-deformation ($n \geq 2$) and $G - 6 \Rightarrow G - 5$ is called $D_{n+2}^-$-deformation. They are generally called $D_{n,2}$-deformations. Here we omit denoting the vertices.
(4) \( G-7 \Rightarrow G-8 \) is called \( D_l^{*-} \)-deformation and \( G-8 \Rightarrow G-7 \) is called \( D_l^{*+} \)-deformation. They are generally called \( D_l^{*-} \)-deformations \( (l \geq 2) \).

(1), (2), (3) and (4) are called DS-deformations for DS-diagrams by the lump. It is known that DS-deformations preserve homeomorphism of a 3-manifold.

The methods of H-cut-transformations are the followings.

1. \( H_a, H_b \)-transformation : \( 1-A \Rightarrow 1-B1 \) (1-B2 resp.) and \( 1-A' \Rightarrow 1-B1' \) (1-B2' resp.)

Step 1. If we deform \( (1-A) \cup (1-A') \) by \( D_{l+2}^{-} \)-deformation \( (l \geq 2) \) shown as the dotted lines, then we get deformed diagrams \( (1-C) \cup (1-C') \) without dotted lines. In those deformations, we should put on new labels to some vertices (edges, faces resp.) satisfying the conditions of Prop. 1, but we sometimes use the former labels, for convenience' sake. In the case of \( l = 1 \), applying \( D_3^{*+} \)-deformation to \( (1-A) \cup (1-A') \), we get \( (1-C) \cup (1-C') \) when \( l = 1 \).
1–C

Remark 4. Neither 1–C nor 1–C' is H-cut-diagram but (1–C) ∪ (1–C') gives a presentation of $M^3$ from a viewpoint of DS-diagram.
Step 2. Next we deform $1-C \cup (1-C')$ by $D_{p,2}^+$-deformation ($p \geq 2$), $D_{3}^-$-deformation ($p = 1$), and $D_{0}^-$-deformation ($p = 0$), shown as the thick dotted lines. By changing the labels, we get $(1-B1) \cup (1-B1')$.

Similarly the transformation from $1-A$ ($1-A'$ resp.) to $1-B2$ ($1-B2'$ resp.) is obtained.

2. $H_{0}$, $H_{0}$-transformation: $1-A \Rightarrow 1-B3$ and $1-A' \Rightarrow 1-B3'$

Step 1. Again we obtain $(1-C) \cup (1-C')$ from $(1-A) \cup (1-A')$ without dotted lines and circles.
Step 2. Next if we deform \((1-C) \cup (1-C')\) by \(D_3^+\)-deformation shown as the dotted lines and circles, then we get \((1-B3) \cup (1-B3')\).

\(H_5, H_6\)-transformation: \(2-A \Rightarrow 2-B\) and \(2-A' \Rightarrow 2-B'\)

Step 1. If we deform \((2-A) \cup (2-A')\) by \(D_{l+2}\)-deformation \((l \geq 2)\) shown as the dotted lines, then we get \((2-C) \cup (2-C')\) without dotted lines and circles. In the case of \(l = 1\), applying \(D_3^+\)-deformation to \((2-A) \cup (2-A')\), we get \((2-C) \cup (2-C')\) when \(l = 1\).
Step 2. Next if we deform (2−C) U (2−C′) by \( D^*_t \)-deformation shown as the dotted lines, and circles, then we get (2−B) U (2−B′).

\[ i (i = 3, 4, 5). \quad i−A \Rightarrow i−B \quad \text{and} \quad i−A′ \Rightarrow i−B′ \]

If we deform (i−A) U (i−A′) by \( D^*_t \)-deformation when \( i = 3 \) and \( D^*_t \)-deformation when \( i = 4, 5 \) shown as the dotted lines, then we get (i−B) U (i−B′).

The above transformations hold the conditions of Prop. 1 and they preserve homeomorphism of a 3-manifold because the DS-deformations preserve homeomorphism.

5. Proofs of Theorem 1

Proofs of (1) and (1′). Let \( D_l(\partial D_l = m_l) \), \( D_j(\partial D_j = m_j) \) be meridian disks of \( U_1−A \), a part of the H-handlebody \( U \). Let \( D_l(\partial D_l = l_h), \cdots, D_p(\partial D_p = l_p), D_l(\partial D_l = l_h), \cdots, D_q(\partial D_q = l_q) \), \( D_p(\partial D_p = l_p) \), \( \cdots, D_q(\partial D_q = l_q) \) be meridian disks of \( V_1−A' \), a part of the H-handlebody \( V \). By cutting off \( U_1−A \) (\( V_1−A' \) resp.) at those meridian disks, \( 1−A \) (\( 1−A' \) resp.) is obtained. Similarly, \( 1−B1 \) (\( 1−B1' \) resp.) is obtained from \( U_1−B1 \) (\( V_1−B1' \) resp.). Therefore, the transformation from \( U_1−A \) (\( V_1−A' \) resp.) into \( U_1−B1 \) (\( V_1−B1' \) resp.) corresponds to the transformation from \( 1−A \) (\( 1−A' \) resp.) into \( 1−B1 \) (\( 1−B1' \) resp.).

Gluing together the two disks put on the same labels, \( \{ D_l, D_j \} \), \( \{ D_p, D_q \} \) in \( 1−B2 \), a Heegaard diagram is obtained. Let the diagram be \( U_1−B2 \). Therefore there exists a transformation from \( U_1−A \) into \( U_1−B2 \) corresponding to that from \( 1−A \) into \( 1−B2 \). Similarly the dual diagram is obtained from \( 1−B2' \). Let the diagram be \( V_1−B2' \). Then there exists a transformation from \( V_1−A' \) into \( V_1−B2' \) corresponding to that from \( 1−A' \) into \( 1−B2' \).

Proofs of (2) and (2′). The H-cut-diagrams derived from \( U_1−A \) (\( V_1−A' \) resp.) and \( U_1−B3 \) (\( V_1−B3' \) resp.) are \( 1−A \) (\( 1−A' \) resp.), \( 1−B3 \) (\( 1−B3' \) resp.), respectively. Therefore there is a transformation from \( U_1−A \) (\( V_1−A' \) resp.) into \( U_1−B3 \) (\( V_1−B3' \) resp.) corresponding to that from \( 1−A \) (\( 1−A' \) resp.) into \( 1−B3 \) (\( 1−B3' \) resp.).

Proofs of (i) and (i′) (\( i = 3, 4, 5, 6 \)). The H-cut-diagram of \( U(i−1)−A, V(i−1)−A' \), \( U(i−1)−B, V(i−1)−B' \) is \( (i−1)−A, (i−1)−A', (i−1)−B, (i−1)−B' \), respectively. Therefore, the transformation from \( U(i−1)−A \) (\( V(i−1)−A' \) resp.) into \( U(i−1)−B \) (\( V(i−1)−B' \) resp.) corresponds to that from \( (i−1)−A \) (\( (i−1)−A' \) resp.) into \( (i−1)−B \) (\( (i−1)−B' \) resp.).

6. Operations for the transformations of H-diagrams and H-cut-diagrams

The operations used for the transformations of H-cut-diagrams are DS-deformations for the DS-diagrams. Therefore we are interested in what kinds of operations are used for the transformations of H-diagrams. From now on we illustrate methods of the transformations of H-diagrams using the operations concerning handlebodies given in Def. 3.

(1) Transformation from \( U_1−A \) into \( U_1−B1 \) is a handle sliding of \( h_m \) about \( h_m' \) along the directions of the longitudes \( \{ l_i, \cdots, l_q \} \) in \( \partial(B_i^{pq} + h_m) \), i.e., the transformation from \( 1−A \) into \( 1−B1 \) of H-cut-diagram corresponds to the handle sliding of \( h_m \) about \( h_m' \) of H-diagram.

This transformation is also obtained by the following operations; first if we carry out the
handles combining with $h_{m \alpha}$ and $h_{m \beta}$, then we get a following figure U1–C. The deformed part is decomposed into four balls $B_i \cup B_j \cup (B_{ij})_u \cup (B_{ij})_b$.

Second by an ambient isotopy of $(B_{ij})_b$, keeping $D_{ij} \times 0$ fixed, and sliding $D_{ij} \times 1$ along the direction of the line, then we get two new handles. Third by changing the labels, U1–B1 is obtained.

We can also obtain U1–B2 by a handle sliding of $h_{m \beta}$ about $h_{m \alpha}$.

(1') The dual transformation from V1–A' into V1–B1' is the longitudes combining with $m_i$ and $m_j$ in V1–A' and the longitude changing of $m_i$.

We can also obtain V1–B2' by the longitude changing of $m_i$.

(2') Transformation from U1–A into U1–B3 is the following operations. First we construct U1–C from U1–A. Second we carry out a handlebody cutting by a disk; parts of $l_{ij}$ and $l_{ij}$ that run side by side from $\partial D_i \times 0$ to $\partial D_i \times 1$ are two segments. A belt that contains those segments as its boundary also runs on the same plane. Let the belt be $Bt$. Let $Bt_i$, $Bt_j$, $Bt_{ij}$, respectively be a part of $Bt$ that runs from $\partial D_i \times 0$ to $m_i$, from $m_i$ to $\partial D_i \times 1$, between $m_i$ and $m_j$. Then $Bt_i \cup Bt_j \cup Bt_{ij}$ represents $Bt$. Let $N(Bt_i, B_i) = N(Bt_j, B_j)$, $N(Bt_{ij}, (B_{ij})_u)$ be a regular neighborhood of $Bt_i$, $Bt_j$, $Bt_{ij}$ in $B_i$, $B_j$, $(B_{ij})_u$, respectively. Let $D_i = \text{Cl}(\partial N(Bt_i, B_i) \cap \text{Int}(B_i))$, $D_j = \text{Cl}(\partial N(Bt_j, B_j) \cap \text{Int}(B_j))$ and $D_{ij} = \text{Cl}(\partial N(Bt_{ij}, (B_{ij})_u) \cap \text{Int}((B_{ij})_u))$, respectively. Then $D_i$, $D_j$, $D_{ij}$ become disks and $D_i \cup D_{ij} \cup D_j$ also becomes a disk. Let $D = D_i \cup D_{ij} \cup D_j$. Then by cutting off $U$ at $D$, a new handle $h_{\alpha \beta}$ appears.

Conversely transformation from U1–B3 into U1–A is obtained by the handle cutting of $h_{\alpha \beta}$ at the meridian disk $D_i' \cap (\partial D_i' = \alpha)$ and the inverse of the handles combining with $h_{m \alpha}$ and $h_{m \beta}$.

(2') The dual transformation from V1–A' into V1–B3' is the longitudes combining with $m_i$ and $m_j$ and longitude $m_{\alpha \beta}$ added to $m_i$ and $m_j$. This transformation is also obtained by a handlebody cutting of $V$ by a disk or an annulus so that the longitudes run as shown in V1–B3'.
Conversely transformation from V1–B3' into V1–A' is obtained by the handle cutting of \( h_{i,A}' \) at the meridian disk \( D_{i,A} (\partial D_{i,A} = l_A) \) and the inverse the longitudes combining with \( m_i \) and \( m_j \).

(3) Transformation from U2–A into U2–B is the handle gluing of \( h_{n_i,1} \) so that the parts of longitudes \( \{ l_i, \ldots, l_n \} \) between the feet of \( h_{n_i} \) in \( \partial B_v^3 \) shrink, and the handle added to \( U \) or handlebody cutting of \( U \) by a disk so that the longitudes run shown in U2–B.

Conversely transformation from U2–B into U2–A is obtained by the handle cutting of \( h_{i,A}' \) at the meridian disk \( D_{i,A} (\partial D_{i,A} = l_A) \) and the inverse of the handle gluing of \( h_{n_i,1} \).

(3') Transformation from V2–A' into V2–B' is the longitude gluing of \( m_i \) and longitude adding to \( m_i \).

This transformation is also obtained by the handlebody cutting of \( V \) by a disk or an annulus so that the longitudes run as shown in V2–B'.

Conversely transformation from V2–B' into V2–A' is obtained by the handle cutting of \( h_{i,A}' \) at the meridian disk \( D_{i,A} (\partial D_{i,A} = l_A) \) and the inverse of the longitude gluing of \( m_i \).

(4) and (4'). Transformations from (U3−A) ∪ (V3−A') into (U3−B) ∪ (V3−B') is a handle cutting of \( h_m \) at the meridian disk \( D \ (\partial D = m) \) of \( U \).

(5) and (5'). Transformation from V4−A' into V4−B' is the following operations. By handle crushing at \( D_{i,A}' \), the longitudes \( m_i \) and \( m_j \) meet in the center point \( p \) of \( D_i \) so that they run as a cross in the neighborhood of \( p \). Next by cutting the deformed handle at \( p \), a new circle is obtained. We put the original label \( m_i \) on it and reorienting \( m_i \). Then V4−B' is obtained. Transformation from U4−A into U4−B is obtained corresponding to the above transformation.

(6) (6') resp.) Transformation from U5−A (V5−A' resp.) into U5−B (V5−B' resp.) is the longitude (meridian resp.) \( l_i \) sliding of \( \partial U \ (\partial V \ resp.) \).

**Definition 10.** The above transformations will be generally called the \( H \)-transformations for Heegaard diagrams.

Next we will devote the following table as the second theorem that describes the operations used for the transformations of H-diagrams and DS-deformations used for those of H-cut-diagrams.
### Theorem 2.

<table>
<thead>
<tr>
<th>Transformation of H-diagram</th>
<th>Operation for H-diagram</th>
<th>Transformation of H-cut-diagram</th>
<th>DS-deformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>U1–A ⇒ U1–B1</td>
<td><em>handle sliding</em> or handles combining and <em>(B_i)</em> sliding</td>
<td>1–A ⇒ 1–B1 and <em>(1–B2)</em></td>
<td><em>D_t^{+}, D_{t+2}^{+}, D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>V1–A′ ⇒ V1–B1′</td>
<td>longitudes combining and longitude changing</td>
<td>1–A′ ⇒ 1–B1′ and <em>(1–B2′)</em></td>
<td><em>D_{t+2}^{+}, D_{t+2}^{+}, D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>U1–A ⇒ U1–B3</td>
<td>handles combining handlebody cutting by a disk or handle adding</td>
<td>1–A ⇒ 1–B3</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>V1–A′ ⇒ V1–B3′</td>
<td>longitudes combining to m_i and m_j handlebody cutting by an annulus or a disk</td>
<td>1–A′ ⇒ 1–B3′</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>U2–A ⇒ U2–B</td>
<td>handle gluing handlebody cutting by a disk</td>
<td>2–A ⇒ 2–B</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>V2–A′ ⇒ V2–B′</td>
<td>longitude gluing to m_i handlebody cutting by an annulus or a disk</td>
<td>2–A′ ⇒ 2–B′</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>U3–A ⇒ U3–B</td>
<td>handle cutting</td>
<td>3–A ⇒ 3–B</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>V3–A′ ⇒ V3–B′</td>
<td></td>
<td>3–A′ ⇒ 3–B′</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>U4–A ⇒ U4–B</td>
<td>handle crushing and cutting</td>
<td>4–A ⇒ 4–B</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>V4–A′ ⇒ V4–B′</td>
<td></td>
<td>4–A′ ⇒ 4–B′</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>U5–A ⇒ U5–B</td>
<td>longitude sliding</td>
<td>5–A ⇒ 5–B</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
<tr>
<td>V5–A′ ⇒ V5–B′</td>
<td>meridian sliding</td>
<td>5–A′ ⇒ 5–B′</td>
<td><em>D_{t+2}^{+} → D_{t+2}^{+} (D_{t+2}^{+})</em></td>
</tr>
</tbody>
</table>

*Proofs.* Notation $D_{t+2}^{+}$ (resp.) → $D_{t+2}^{+}$ denotes that applying $D_{t+2}^{+}$-deformation after $D_{t+2}^{+}$ (resp.)-deformation to (1–A) U (1–A′), we obtain (1–B1) U (1–B1′) via (1–C) U (1–C′). We see that the half way of the process of the handle sliding of $m_i$, about $m_{ij}$, i.e., two operations of handle combining with $m_i$ and $m_{ij}$, and longitudes combining with $m_i$ and $m_{ij}$.
correspond to the $D_{i+2}^{-}$ ($D_2^+$ resp.-)deformation. Next $D_{p+2}^{+}$ ($D_3^-$, $D_2^{-}$ resp.)-deformation gives the transformation from $(1-C) \cup (1-C')$ into $(1-B1) \cup (1-B1')$, i.e., $D_{m_i}$ jumps out of $D_{m_i}$ in $1-C$. The latter part of the handle sliding, and the longitude changing of $m_i$ correspond to $D_{p+2}^{+}$ ($D_3^{-}$, $D_2^{-}$ resp.)-deformation. □

$D_{i+2}^{-}$ ($D_3^+$ resp.-) $\rightarrow$ $D_3^+$ gives the transformations from $(1-A) \cup (1-A')$ into $(1-B3) \cup (1-B3')$. Again we obtain $(1-C) \cup (1-C')$ from $(1-A) \cup (1-A')$ by $D_{i+2}^{-}$ ($D_3^+$ resp.)-deformation. Next we obtain $(1-B3) \cup (1-B3')$ from $(1-C) \cup (1-C')$ by $D_3^+$-deformation. Therefore we see that the handlebody cutting by a disk and the longitude $m_i$ added to $m_i$ and $m_j$ correspond to $D_3^+$-deformation. □

The second $D_{i+2}^{-}$ ($D_3^+$ resp.-) $\rightarrow$ $D_3^+$ gives the transformations from $(2-A) \cup (2-A')$ into $(2-B) \cup (2-B')$. The handle gluing of $h_{m_i}$ and the longitude gluing of $m_i$ correspond to $D_{i+2}^{-}$ ($D_3^+$ resp.)-deformation. The handle added to $U$ or handlebody cutting of $U$ by a disk, and longitude adding to $m_i$ correspond to $D_3^+$-deformation. □

It is not so difficult to see that $D_i^+$ and the two $D_i^-$-deformations correspond to the left-side operations for H-diagrams. Therefore, we omit the remaining proofs in the table. □

We have obtained the operations of the transformations of H-diagrams corresponding to those of H-cut-diagrams. However it is very difficult to transform H-diagrams as can be seen from the following examples 1 and 2. Therefore, it will be desirable to transform the H-cut-diagrams by using the $DS$-deformations instead of H-diagrams such as done in example 3. Some examples in [8] will be helpful to master the transformations of H-cut-diagrams.

7. Some examples

Here we give three examples promised at the Introduction.

**Example 1.** The following figures $(U; m, l, 12)$, $(V; l, m, 12)$ are genus 2 H-diagrams of $(U, V, F)$ of the 3-sphere $S^3$. The number 12 of $(U; m, l, 12)$ ($(V; l, m, 12)$ resp.) indicates the cross point number of $\{m\} \cap \{l\}$. Here $Cl(\partial B^3 \cup_{(\{D_i \times 0, D_i \times 1\})} (Cl(\partial B^3 \cup_{(\{D'_i \times 0, D'_i \times 1\})}$ resp.) is the H-cut-diagram $G(m, l)$ ($G(l, m)$ resp.) of $(U; m, l, 12)$ ($(V; l, m, 12)$ resp.). $G(m, l)$ ($G(l, m)$ resp.) has two waves. By the definition of a wave, note that if $G(m, l)$ has waves $\{w_1, w_2, \ldots, w_n\}$, then $G(l, m)$ has the same waves as $G(m, l)$.

From now on, we carry out transformations from $(U; m, l, 12)$, $(V; l, m, 12)$ into the canonical genus 1 H-diagram, respectively.
Tfm \((V; l, m, 12)\) (Transformation of \((V; l, m, 12)\)); if we carry out a handle sliding of \(h_1'\) about \(h_2'\) along \(10(m_1^{-1})5\), a part of longitude \(m_i\) in \(\partial(B_{r,3}+h_2')\), then we have the following H-diagram \((V; l, m, 7)\).
Tfm $(U; m, l, 12)$; if we carry out longitudes combining with $l_1$ and $l_2$, and longitude changing of $l_2$ at $5(m_1)10$ in $\partial D_1 \times 0$, then $(U; m, l, 7)$ is obtained.

$(U; m, l, 7)$

$(V; l, m, 7)$
Tfm \((V; l, m, 7)\); by a handle sliding of \(h'_1\) about \(h'_1\) along \(4(m_i^{-1})10\) in \(\partial(B_v^9 + h'_1)\), \((V; l, m, 5)\) is obtained.

Tfm \((U; m, l, 7)\); by longitudes combining with \(l_i\) and \(l_j\), and longitude changing of \(l_i\) at \(10(m_i)4\) in \(\partial D_i \times 1\), \((U; m, l, 5)\) is obtained.
Tfm \((V ; l, m, 5)\); again by a handle sliding of \(h'_2\) about \(h'_1\) along \(4(m_1^{-1})11\) in \(\partial(B_v + h'_1)\), \((V ; l, m, 3)\) is obtained.

Tfm \((U ; m, l, 5)\); again by longitudes combining with \(l_1\) and \(l_2\), and longitude changing of \(l_1\) at \(11(m_1)4\) in \(\partial D_1 \times 1\), \((U ; m, l, 3)\) is obtained.

\((U ; m, l, 3)\)

Tfm \((U ; m, l, 3)\), \((V ; l, m, 3)\); by cutting off \(h_2\) at the meridian disk \(D_2\) \((\partial D_2 = m_2)\), genus 1 H-diagrams \((U ; m, l, 1)\), \((V ; l, m, 1)\) are obtained. Those are the special cases of H-diagrams in Fig. 5 when \(n = 1\).
Example 2. The following figure \((U; m, l, 7)\) is a genus 1 H-diagram of \((U, V, F)\) of the lens space \(L(7, 2)\). \((V; l, m, 7)\) is the dual diagram. It is known that \(L(7, 2)\) is homeomorphic to \(L(7, 4)\). Therefore we transform \(L(7, 2)\) into \(L(7, 4)\).
Tfm \((U; m, l, 7), (V; l, m, 7)\); dotted lines and small dotted half circles in \((U; m, l, 7)\) \((V; l, m, 7)\) resp.) indicate a properly embedded 2-disk \(D\) \((D') \text{ resp.}) in \(U\) \((V\) resp.). By cutting off \(U\) \((V\) resp.) at \(D\) \((D'\) resp.), genus 2 H-diagram \((U; m, l, 8)\) \((V; l, m, 8)\) resp.) is obtained.

\[h_1 \quad m_1\]

\[l_1, l_2, l_3, l_4, \ldots\]

\[h_2, m_2, m_3, m_4, \ldots\]

\((U; m, l, 8)\)

\[h_1' \quad m_1\]

\[l_1, l_2, l_3, l_4, \ldots\]

\[h_2', m_2, m_3, m_4, \ldots\]

\((V; l, m, 8)\)
Tfm \((U; m, l, 8)\); by handle sliding of \(h_1\) about \(h_2\) along \(7(l_1^{-1})2\) in \(\partial(B_2 + h_2)\), \((U; m, l, 9)\) is obtained.

Tfm \((V; l, m, 8)\); by longitudes combining with \(m_1\) and \(m_2\), and longitude changing of \(m_2\) at \(2(l_1)7\) in \(\partial D_1 \times 1\), \((V; l, m, 9)\) is obtained.
Here we rewrite a drawing $V$ over, for convenience' sake.

\[ (V; l, m, 9) \]

$\text{Tfm } (V; l, m, 9); \text{ by handle sliding of } h_1' \text{ about } h_2' \text{ along } l(m_2^{-1})9 \in \partial(B_v^3 + h_2'), (V; l, m, 9) \text{ is obtained.}$

$\text{Tfm } (U; m, l, 9); \text{ by longitudes combining with } l_1 \text{ and } l_2, \text{ and longitude changing of } l_2 \text{ at } 9(m_2)1 \text{ in } \partial D_3 \times 1, (U; m, l, 9) \text{ is obtained.}$
Tfm \((V; l, m, 9)\); again by handle sliding of \(h_1'\) about \(h_2'\) along \(1(m_2^{-1})3\) in \(\partial(Bv^3 + h_2')\), \((V; l, m, 9)\) is obtained.
Tfm \((U; m, l, 9)\); again by longitudes combining with \(l_1\) and \(l_2\), and longitude changing of \(l_2\) at \(3(m_2)1\) in \(\partial D_2 \times 0\), \((U; m, l, 9)\) is obtained.

\[
\begin{align*}
&\quad \begin{array}{c}
\text{h}_1 \\
l_2
\end{array} \quad \begin{array}{c}
m_1
\end{array} \\
\partial D_2 \times 1
\end{align*}
\]

\((U; m, l, 9)\)

Tfm \((V; l, m, 9)\); again by handle sliding of \(h'_1\) about \(h'_2\) along \(1(m_2)2\) in \(\partial (B_1^3 + h'_2)\), \((V; l, m, 9)\) is obtained.

Tfm \((U; m, l, 9)\); again by longitudes combining with \(l_1\) and \(l_2\), and longitude changing of \(l_2\) at \(1(m_2)2\) in \(\partial D_2 \times 1\), \((U; m, l, 11)\) is obtained.
Tfm \((U; m, l, 11), (V; l, m, 11)\); by cutting off \(h_2\) at the meridian disk \(D_2 (\partial D_2 = m_2)\), genus 1 diagrams \((U; m, l, 7), (V; l, m, 7)\) of the lens space \(L(7, 4)\) are obtained.
Therefore we have transformed such case from $L(7, 2)$ into $L(7, 4)$.


The following figures $(U ; m, l, 11)$ $\cup$ $(V ; l, m, 11)$ are genus 4 H-diagrams of $(U, V, F)$ of $S^3$ obtained in [9]. Those H-cut-diagrams $G_1(m, l, 11)$ $\cup$ $G_1(l, m, 11)$ have no-waves, respectively. Therefore, it is also a counter example. Here our method of construction is different from them.
\( (U; m, l, 11) \)
\[(V; l, m, 11)\]

\[G_i(m, l, 11)\]
The transformations of the H-diagrams are not easy as seen in Ex. 1, 2. Therefore we carry out transformations from no-waves H-cut-diagrams $G_1(m, l, 11) \cup G_1(l, m, 11)$ to waves ones corresponding to those of H-diagrams $(U; m, l, 11) \cup (V; l, m, 11)$.

Tfm $G_1(m, l, 11), G_1(l, m, 11)$; in $G_1(m, l, 11) \cup G_1(l, m, 11)$ there are two kinds of dotted lines and circles; the fine dotted lines, circles and thick dotted lines. First we draw the three fine dotted lines and two circles in $G_1(m, l, 11) \cup G_1(l, m, 11)$. Second under those dotted lines and circles, we deform the diagrams so that parts of $G_1(m, l, 11) \cup G_1(l, m, 11)$ become the thick dotted lines. Then we have waves diagrams $G_2(m, l, 13) \cup G_2(l, m, 13)$.

The transformation from $G_1(l, m, 11)$ into $G_2(l, m, 13)$ corresponds to the handle sliding $h_1$ about $h_2$ along $l_2^{-1}2$. The transformation from $G_1(m, l, 11)$ into $G_2(m, l, 13)$ corresponds to the longitudes combining with $l_1$ and $l_2$, and longitude changing of $l_2$ at $2l_2$. 
In this way, by handle sliding, waves diagram is obtained. Further $G_t(m, l, 13)$, $G_t(l, m, 13)$ are transformed into the canonical genus 1 diagrams, respectively.

A complicated Ochiai's is also transformed into the canonical one in [8]. Therefore, we are interested in the reason the above H-(cut)-diagrams of the 3-sphere become the canonical ones.

Henceforth we will need go a step further for the study of the transformations from (1−A) U (1−A′) into (1−B1) U (1−B1′) for the 3-sphere.
References

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