TRANSFORMATIONS OF THE FUNDAMENTAL GROUPS CORRESPONDING TO THOSE OF HEEGAARD DIAGRAMS BY THE BAND MOVES

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1. Introduction

H. Zieschang obtains an important result concerning the band move for a handlebody (§3 Theorem 4). We will give transformations by the band move for basic Heegaard diagrams (§3 Theorem 5). From the Theorems 4 and 5, we get more developed transformations for Heegaard diagrams (§3 Theorem 6). F. Waldhausen obtains an important result concerning the equivalent of Heegaard splittings for the 3-sphere $S^3$ (§3 Theorem 7). Moreover, from the Theorems 6 and 7, we get a distinguishing feature of transformations of Heegaard diagrams for $S^3$ (§3 Theorem 8).

§4 deals with the main theme that the transformation by a band move of the Heegaard diagram and that of the fundamental group by an algebra calculation (replacements or substitution) correspond to 1 to 1 (Theorem 9). Moreover, a reduction of the fundamental groups for $S^3$ is obtained (Theorem 10).

Everything in this paper, we will be considering the piecewise linear point of view. $\partial X$, $Int(X)$, $Cl(X)$ indicates the boundary, interior, closure of a point set $X$, respectively. Hereafter, notation $M^3$ denotes a closed, connected orientable 3-manifold unless otherwise stated.

2. Preliminaries

In this section we give definitions, basic Theorems 1 and 2, and so forth.

We begin with a definition of a handlebody.

**Definition 1.** Let $\{D_1, \cdots, D_n\}$ be mutually disjointed 2-disks and $h_i = D_i \times [0, 1]$ ($i = 1, \cdots, n$). A handlebody $H$ of genus $n$ is a 3-ball (cube) $B^3$ with $n$ handles $\{h_i\}$ so that the result of attaching $h_i$ with homeomorphisms throws $2n$ disks $D_i \times 0$, $D_i \times 1$ onto $2n$ disjointed 2-disks on $\partial B^3$. $H$ is represented as $B^3 + \bigcup h_i$ where $B^3 \cap h_i = \partial B^3 \cap \partial h_i = \{D_i \times 0, D_i \times 1\}$. A handlebody $H$ of genus $n$ is also called as a solid torus of genus $n$.

We note that $\partial H$ is an orientable or nonorientable closed surface of Euler characteristic.
2−2n according as \( H \) is orientable or nonorientable.

**Definition 2.** Let \( H \) be a genus \( n \) handlebody and \( \{D_i\} (i = 1, \cdots, n) \), mutually disjointed properly embedded 2-disks in \( H \). If the \( Cl(H - \{D_1 \cup \cdots \cup D_n\}) \) becomes 3-ball, then the collection \( \{D_i\} (i = 1, \cdots, n) \) is called a complete system of meridian disks of \( H \) and \( \{\partial D_i\} \) a complete system of meridian circles of \( \partial H \).

Note that \( \{D_1, \cdots, D_n\} \) cuts \( \partial H \) into 2-sphere with \( 2n \) holes.

**Definition 3.** Let \( H \) be an orientable genus \( n(\geq 2) \) handlebody with the same presentation as in Def. 1.

1. Fig. 2–1 shows two handles \( h_i \) and \( h_j \) of \( H \). By an ambient isotopy of \( H \), keeping \( D_i \times 0 \) fixed, and sliding \( D_i \times 1 \) along the direction of the line in \( \partial(B^3 + h_i) \), \( h_i \) goes over the \( h_j \) and turns back to the first place. This operation is called a handle sliding of \( h_i \) about \( h_j \).

2. Let \( \{D_i\} (i = 1, \cdots, n) \) be a complete system of meridian disks of \( H \) and \( \{m_i\} (m_i = \partial D_i) \) a complete system of meridian circles of \( \partial H \). Let \( \alpha \) be an arc on \( \partial H \) that joins two chosen meridians \( m_i \), \( m_j \) and \( Int(\alpha) \cap \{m_i \cup m_j\} = \phi \). See Fig. 2–2. Let \( N(m_i \cup \alpha \cup m_j, \partial H) \) be a regular neighborhood of \( m_i \cup \alpha \cup m_j \) on \( \partial H \). \( \partial N \) consist of three circles. Out of the three circles, two are isotopic to \( m_i \), \( m_j \) and then the remainder is not isotopic to them. Let the notation of remainder be \( m_{ij} \). \( m_{ij} \) is called a band sum of \( m_i \) and \( m_j \) (with respect to the band \( \alpha \)). It has also the very pleasant property that bounds a disk and it is homeomorphic to \( D_i \) and \( D_j \). Changing the label \( m_{ij} \) into \( m_i \) (\( m_j \) resp.) is called a band move of \( m_i \) (\( m_j \) resp.).

3. Let \( \bar{D}_i, \bar{D}_j \) be a disk in the foot of \( \partial h_i, \partial h_j \) shown in Fig. 2–3, respectively. Gluing together \( h_i \) and \( h_j \) by an orientation-reversing homeomorphism \( f: \bar{D}_i \rightarrow \bar{D}_j \), a handlebody with the deformed part, the figure 3-shape turned to \( \pi/4 \) radians is obtained (see §3 U1–C). This operation is called as handles combining with \( h_i \) and \( h_j \).
Definition 4. A closed, connected 3-manifold $M^3$ is represented with a union of two handlebodies $H_1, H_2$ in $M^3$; $M^3 = H_1 \cup H_2$ so that $H_1 \cap H_2 = \partial H_1 \cap \partial H_2 = \partial H_1 = \partial H_2$. $\partial H_1$ ($= \partial H_2$) is a closed surface of genus $n(\geq 1)$. Let the surface be $F$. $H_1$ ($H_2$ resp.) and $F$ are orientable or nonorientable according as $M^3$ is orientable or nonorientable. A triplet $(H_1, H_2, F)$ or $M^3 = H_1 \cup H_2$ is called a Heegaard splitting (H-splitting) of $M^3$ with genus $n$ and $H_1$ ($H_2$ resp.), a Heegaard-handlebody (H-handlebody). $F$ is called a Heegaard-surface (H-surface) and the integer $n(\geq 1)$, Heegaard genus (H-genus). Let $U$ and $V$ be disjointed handlebodies with the same genus. Let $f : U \to V$ be a homeomorphism so that $f|_{\partial U} : \partial U \to \partial V$ is an orientation-reversing homeomorphism. Gluing together $\partial U$ of $U$ and $\partial V$ of $V$ by $f$, we get $M^3$. Then $M^3$ is denoted as $(M^3; U, V, f)$ or $M^3 = U \cup V$. It is called a genus $n$ H-splitting of $M^3$ concerning $f$.

In $(M^3; U, V, f)$, by replacing $f^{-1}(V)$ with $V$, one can regard $(M^3; U, V, f)$ as $(U, V, F)$ of $M^3$.

Theorem 1. Let $M^3 = H_1 \cup H_2$ and $M'^3 = H'_1 \cup H'_2$ be two H-splittings with the same genus. Suppose that there exist homeomorphisms $f : H_1 \to H'_1$ and $g : H_2 \to H'_2$ so that the right side diagram becomes commutative. Then $M^3$ is homeomorphic to $M'^3$.

Proof. Suppose that $h : M^3 \to M'^3$ is a homeomorphism so that $h|_{\partial H_i} = f$ and $h|_{\partial H_2} = g$.

Then by the above commutative diagram, $h$ is well-defined. □

Theorem 2. Let $M^3 = H_1 \cup H_2$ be a genus $n$ H-splitting and $\psi : H_1 \to H_1$ a homeomorphism. Let $M'^3 = H_1 \bigcup_{\phi \psi(\partial H_1)} H_2$. Then $M'^3$ is homeomorphic to $M^3$.

Proof. Let the elliptical character $id.$ be an identification map of $\partial H_2$. Then the right side diagram becomes commutative. Hence by the Theorem 1, $M'^3$ is homeomorphic to $M^3$. □

By the above Theorem, we can apply the handle sliding and handles combining to H-splitting to examine the changing of $M^3$.

Definition 5. Let $(H_1, H_2, F)$ and $(H'_1, H'_2, F')$ be H-splittings of $M^3$ with the same genus. If there exists a homeomorphism $f : M^3 \to M^3$ so that $f(F) = F'$, then $(H_1, H_2, F)$ and $(H'_1, H'_2, F')$ are called equivalent.

Definition 6. Suppose $(H_1, H_2, F)$ is a genus $n(\geq 1)$ H-splitting of $M^3$. Let $(D_1, \ldots, D_n)$,
Let $\{D_1', \cdots, D_n'\}$ be a complete system of meridian disks of $H_1, H_2$, respectively and $\{m\} = \{m_1, \cdots, m_n\} = \{\partial D_1, \cdots, \partial D_n\}$, $\{l\} = \{l_1, \cdots, l_n\} = \{\partial D_1', \cdots, \partial D_n'\}$. Then $(H_1 ; m, l) ((H_2 ; l, m) \text{ resp.})$ is called a genus $n$ Heegaard diagram (H-diagram) associated with $(H_1, H_2, F)$. $(m, l)(\{l, m\} \text{ resp.})$ are called meridian-longitude systems of $(H_1 ; m, l) ((H_2 ; l, m) \text { resp.})$.

By an ambient isotopy of $H$, a genus $n(\geq 1)$ handlebody $H$ is deformed such as shown in Fig. 2-4. This shows a genus $n$ H-diagram $(H_i ; m, l)$ of the 3-sphere. It is called a canonical genus $n$ H-diagram.

Let $(H_1 ; m_1, \cdots, m_n, l, \cdots, l_n)$ be a genus $n$ H-diagram associated with $(H_1, H_2, F)$ of $M^3$. We may assume that $(m_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)$ consists at most of finite points (by an argument of general position).

**Definition 7.** The number of finite points of $\{m\} \cap \{l\} = (m_1 \cup \cdots \cup m_n) \cap (l_1 \cup \cdots \cup l_n)$ is called a cross point number with $(H_1 ; m, l)$ or $(H_2 ; l, m)$.

### 3. Transformations of Heegaard diagrams

We begin with an obvious Proposition.

**Proposition 3.** Let Fig. 2-5 be a part of H-diagram $(U ; m, l)$. The longitude $l_i$ crosses the meridian $m_i$, turns back to $m_i$, and crosses $m_i$, again. Then, there exist a transformation of $(U ; m, l)$ so that a part of $l_i$ deforms to the dotted line and it does not cross $m_i$. It does not change the H-genus but decreases the cross point number, as many as 2.

**Definition 8.** The above transformation is called a cancelling for the H-diagram.

If the diagram like Fig. 2-5 appears, then we always do the above correction.

From now, we state Zieschang's result and give transformations by the band move for basic H-diagrams after that.

**Theorem 4.** Let $H$ be a genus $n(\geq 2)$ handlebody. Then any two complete systems of meridian circles of $\partial H$ transform each other under a finite sequence of band moves (Zieschang [2]).
Let the following figure U1–A be a part of H-diagram \((U; m, l)\). The longitudes \(\{l_{i1}, \ldots, l_{il}\}\) \((l \geq 0)\) drawn heavily go around side by side on the two handles \(h_i\) and \(h_j\). The longitudes \(\{l_{i1}, \ldots, l_{il}\}, \{l_{i1}, \ldots, l_{il}\}\) go around on \(h_i, h_j\), respectively. It shows the general case that longitudes run on handles \(h_i\) and \(h_j\). In a special case that a character \(l\) on the lower right equals to 0, there are not longitudes that run on \(h_i\) and \(h_j\). V1–A’ is the dual part of U1–A. The longitude \(m_i, m_j\) crosses the meridians \(\{l_{i1}, \ldots, l_{il}, l_{i1}, \ldots, l_{il}\}, \{l_{i1}, \ldots, l_{il}, l_{i1}, \ldots, l_{il}\}\), respectively.
The transformation from $U1-A$ into $U1-B$ is obtained by the handle sliding of $h_i$ about $h_j$ along the directions of the longitudes $\{l_{ij}, \ldots, l_{ij}\}$ in $\partial(B^3 + h_i)$. In $U1-B$, $\{l_{ij}, \ldots, l_{ij}\}$ go around on $h_i$ (not on $h_j$), $\{l_{ij}, \ldots, l_{ij}\}$ go around on both the $h_i$ and $h_j$, and $\{l_{ij}, \ldots, l_{ij}\}$ do not change the way of running. The dual transformation from $V1-A'$ into $V1-B'$ is obtained by a band move: each meridian $l_{ij}$ is cut into two segments by the two longitudes $m_i$ and $m_j$. Let the shorter segment be $\alpha$. From $m_i \cup \alpha \cup m_j$, we may construct a band sum $m_{ij}$ of $m_i$ and $m_j$, and carry out a band move of $m_{ij}$. By an ambient isotopy and reorienting $m_j$, $V1-B'$ is obtained. In $V1-B'$, $m_i$ does not change the way of running, and here $m_j$ comes to cross $\{l_{ij}, \ldots, l_{ij}, l_{ij}, \ldots, l_{ij}\}$.

In the case of $(l = 0)$, if we can draw a band $\beta$ which reaches to $m_j \times 0$ via $m_i \times 1$ as it does not intersect the longitudes, then we can handle sliding $h_i$ about $h_j$ along $\beta + \partial h_j$.

The above handle sliding is regarded as the band move of $m_i$: if we carry out the handles combining with $h_i$ and $h_j$, then $U1-C$ is obtained. The handle $h_i$ drawn heavily attaches under $h_i$. This means that handles combining gives a band move of $m_i$. Next by handle sliding of $h_i$ about $h_i$ along the direction of the line, $U1-B$ is obtained.

In like manners, a handle sliding of $h_j$ about $h_i$ and a band move of $m_i$ are obtained. Hence we have:

**Theorem 5.** The transformation from $U1-A(V1-A'$ resp.) into $U1-B (V1-B'$ resp.) is carry out by a band move of $m_i$. It does not change the $H$-genus but changes the cross point number as many as $|l - p|$.

In $U1-A (V1-A'$ resp.) we can carry out band moves for two meridians and two longitudes in Theorem 5. And band moves to transform $H$-diagrams are only these types.

Applying the Theorem 4 to both the $(U ; m, l)$ and $(V ; l, m)$ in Theorem 5, we have:

**Theorem 6.** If $(U ; m, l) \cup (V ; l, m)$ and $(U ; m', l') \cup (V ; l', m')$ are two sets of the genus $n(\geq 2)$ $H$-diagrams associated with $(U, V, F)$, then $(U ; m, l) ((V ; l, m)$ resp.) is transformed into $(U ; m', l') ((V ; l', m')$ resp.) under a finite sequence of band moves for two meridians and two longitudes.

There is an important result about the equivalent of $H$-splittings for the 3-sphere.

**Theorem 7.** $H$-splittings of the same genus of the 3-sphere are equivalent (Waldhausen [3]).
The above Theorem means that it is made as it chooses meridian-longitude systems \( \{ m_1, \ldots, m_n \}, \{ l_1, \ldots, l_n \} \) suitably in \( \text{H-splitting} \) of the 3-sphere, which satisfy the conditions of \( m_i \cap l_j = \{ \text{a point} \} \) (\( i = j \)) and \( m_i \cap l_i = \emptyset \) (\( i \neq j \)).

From the Theorems 6 and 7, we have:

**Theorem 8.** Any genus \( n(\geq 2) \) \( \text{H-diagram} \) of the 3-sphere is transformed into the canonical one under a finite sequence of band moves for two meridians and two longitudes.

It is not easy to transform \( \text{H-diagrams} \). In [5, 6], we obtain the methods of transformations of Heegaard cut diagrams (\( \text{H-cut-diagrams} \)) corresponding to those of \( \text{H-diagrams} \). They are the ones which have applied DS-deformations (Ikeda, Yamashita, Yokoyama; [10]) to \( \text{H-cut-diagram} \) for \( \text{DS-diagram} \) (Ikeda, Inoue; [7], Ishii [8]). We see that \( D_{\alpha} \)-deformation corresponds to the band move ([6]). In this way, \( \text{DS-diagram} \) and \( \text{H-cut-diagram} \) are closely related (Yamashita [9]).

4. **Transformations of the fundamental groups**

To state our result precisely, we prepare algebra calculations for groups.

**Definition 9.** Let \( \langle a_1, \ldots, a_n \mid r_1 = 1, \ldots, r_m = 1 \rangle \) denotes a presentation of a finitely generated group, where \( a_1, \ldots, a_n \) are generators and relator \( r_i \) is a word in the \( a_i \)'s (\( \varepsilon = \pm 1 \)). We underline to the letters which are operated.

**Replacements letters:** if there are relations \( a_i^*a_j^*w_k = 1 \) (\( k = 1, \ldots, \alpha \)), then replace the generator \( a_i \), letters \( a_i^*a_j^* \) by a new letter \( \tilde{a}_i \) (this becomes a new generator).

**Substitution:** if there are two relations \( w_1a_{i_1}^* \cdots a_{i_{\alpha}}^* = 1 \) and \( w_2a_{i_1}^* \cdots a_{i_{\alpha}}^* = 1 \), where \( a_k \) (\( k = 1, \ldots, \alpha \)) is a generator and \( a_{i_1}^* \cdots a_{i_{\alpha}}^* \) is a common word, then substitute \( a_{i_1}^* \cdots a_{i_{\alpha}}^* = w_{1,2}^{-1} \) for \( w_2a_{i_1}^* \cdots a_{i_{\alpha}}^* = 1 \).

Each above algebra calculation preserves isomorphism of a group.

Let \( (U, V, F) \) be a genus \( n(\geq 1) \) \( \text{H-splitting} \) of \( M^3 \) and \( (U; m, l) \) a \( \text{H-diagram} \) of \( (U, V, F) \). \( \{ m \} = \{ m_1, \ldots, m_n \} \) and \( \{ l \} = \{ l_1, \ldots, l_n \} \) are meridian-longitude systems. Let each \( m_i, l_i \) be oriented. By applying the van Kampen’s Theorem to \( U \cup V \), we may obtain a well-known presentation of a fundamental group \( \pi_1(M^3) \):

\[
\pi_1(M^3) = \langle m_1, \ldots, m_n \mid \hat{l}_1 = 1, \ldots, \hat{l}_n = 1 \rangle \quad (1).
\]

We read that \( m_1, \ldots, m_n \) are regarded as the generators of the meridians \( m_1, \ldots, m_n \) and the relator \( \hat{l}_j \) is a word in the \( m_i^{\varepsilon_j} \)'s obtained by running once around the \( l_j \), i.e., while we take a
turn around \( l_i \) according to the orientation of \( l_i \), we read the label \( m_i \) continuously as \( m_i^{-1} \) (resp.) if \( l_i \) crosses \( m_i \) from the left side (the right side resp.) to the right side (the left side resp.) of \( m_i \). See Fig. 4. In the relator \( \tilde{l}_j \), we may start reading from any \( m_i \) in \( \tilde{l}_j \) because the word \( \tilde{l}_j \) becomes a cyclic word by joining both ends of \( \tilde{l}_j \) and preserving the sequential order of letters in \( \tilde{l}_j \). Hence \( \tilde{l}_j \) is uniquely defined up to cyclic permutations and inversions. A dual presentation from \((V ; l, m)\) of \((U, V, F)\) is also defined in an analogous manner, and is denoted as

\[
\pi_i(M^3) = \langle l_1, \ldots, l_n | \tilde{m}_1 = 1, \ldots, \tilde{m}_n = 1 \rangle \quad (1').
\]

Group (1) is isomorphic to (1') but the presentation (1) is generally different from (1') because meridians and longitudes are switched in \((U ; m, l)\) and \((V ; l, m)\). Hence the forms of relators in (1) and (1') are different generally.

Let a presentation of the fundamental group derived from U1–A, U1–B of \((U ; m, l)\) be (1A), (1B), respectively.

\[
\begin{align*}
\langle m_i, m_j \rangle & : \quad \begin{array}{l}
m_i m_j w_{i k} = 1 \cdots (l_{i k}) \quad (k = 1, \ldots, l) \\
m_k & \quad m_i^{-1} w_{i k} = 1 \cdots (l_{i k}) \quad (k = 1, \ldots, p) \\
(k \neq i, j) & \quad m_j w_{i k} = 1 \cdots (l_{i k}) \quad (k = 1, \ldots, q) \\
r_a & = 1 \quad \text{(relations other than the above)}
\end{array} \\
(1A)
\end{align*}
\]

\[
\begin{align*}
\langle m_i, m_j \rangle & : \quad \begin{array}{l}
m_i m_j w_{i k} = 1 \cdots (l_{i k}) \quad (k = 1, \ldots, l) \\
m_k & \quad m_i m_j^{-1} w_{i k} = 1 \cdots (l_{i k}) \quad (k = 1, \ldots, p) \\
(k \neq i, j) & \quad m_j w_{i k} = 1 \cdots (l_{i k}) \quad (k = 1, \ldots, q) \\
r_a & = 1 \quad \text{(relations other than the above)}
\end{array} \\
(1B)
\end{align*}
\]

Note that the relations \((l_{i k}) \quad (k = 1, \ldots, l)\) in (1A) and (1B), too, do not exist if the longitudes \(l_{i k} \quad (k = 1, \ldots, l)\) do not exist.

operations; in (1A), replace the generator \(m_i\), letters \(m_i \leftrightarrow m_j\) in \((l_{i k})\) by a new letter \(\tilde{m}_i\) (a new generator), we get a presentation that is isomorphic to (1B).

Let a presentation of the fundamental group derived from V1–A', V1–B' of \((V ; l, m)\) be (1A'), (1B'), respectively.

\[
\begin{align*}
\langle l_i, \ldots, l_p \rangle & : \quad \begin{array}{l}
l_i^{-1} \cdots l_p^{-1} l_{i p} \cdots l_{i i} = 1 \cdots (m_i) \\
l_i \cdots l_p l_{i p} \cdots l_{i i} = 1 \cdots (m_j) \\
(k \neq i, j, i j) & \quad r_a' = 1 \quad \text{(relations other than the above)}
\end{array} \\
(1A')
\end{align*}
\]
Operation : in (1A'), by substituting \( l_{ij} \cdots l_{ii} = l_{ii} \cdots l_{ij} \) derived from \((m_i)\) for \( l_{ii} \cdots l_{ij} \) in \((m_j)\), we get (1B').

In like manner, transformations of the fundamental groups corresponding to those of a handle sliding of \( h_i \) about \( h_j \) of \( (U; m, l) \) and a band move of \( m_i \) of \( (V; l, m) \) are obtained.

Hence by gathering the Theorem 6 and considering the above, we have;

**Theorem 9.** Let \((H; a, b)\) and \((H; a', b')\) are the genus \( n(\geq 2) \) \( H \)-diagrams associated with a \( H \)-splitting of \( M^3 \). Then transformations from \((H; a, b)\) into \((H; a', b')\) by a finite sequence of band moves are in 1-1 correspondence with those of the fundamental group derived from \((H; a, b)\) by the replacements and substitutions.

Moreover, from the Theorems 8 and 9 we have;

**Theorem 10.** Any \( H \)-diagram of genus \( n(\geq 2) \) of the 3-sphere and the fundamental group derived from that are reduced to the canonical one and the trivial group by a finite sequence of band moves and corresponding the replacements and substitutions.

We have a lot of examples for the 3-sphere \( S^3 \). Especially, they are ones about waves. The Whitehead [11]-Volodin-Kuznetsov-Fomenko [12] conjecture shows that “all \( H \)-diagrams of \( S^3 \) other than the canonical one have waves without fail.” This is an algorithm for recognizing \( S^3 \) in 3-manifold. In [1], Birman describes that “nobody has succeeded in verifying such an assertion between 1935 and 1977, or producing a counter example.” In 1980, Homma-Ochiai-Takahashi [14] success in the above conjecture if H-genus = 2. But Viro [13], Morikawa [15], Ochiai [17] and the author [6] construct counter examples if H-genus \( \geq 3 \). We can realize the ones of the persons above as examples of Theorem 10.

**References**


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